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С 49

SOME PROBLEMS IN THE FREE-VIBRATION ANALYSIS OF RIGID RECTANGULAR PLATES

ДЕЯКІ ПИТАННЯ В РОЗРАХУНКАХ ВІЛЬНИХ КОЛИВАНЬ ЖОРСТКИХ ПРЯМОКУТНИХ ПЛАСТИН

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Abstract. Some aspects of the free-vibration analysis of rigid rectangular ship structure plates are considered. In particular unusual mode shapes for square plates, to be derived in the FEM analysis of the problem, are explained by using superposition effect. The family of fundamental solutions of the biharmonic equation in free-vibration problem of rigid plates has been obtained. The solutions ware used for the mode shapes analysis of rectangular plates with hinged and clamped sides supports. For the case of fully clamped plates the approximate asymptotic solutions ware proposed too. Comparison of the results for fully clamped plates, derived by using FEM analysis and asymptotic solutions, displayed close results.
Keywords: rectangular plates; free vibration mode shapes; natural frequencies; biharmonic equation; asymptotic solutions.

Анотація. Розглянуто особливості розрахунку власних частот і форм вільних коливань прямокутних жорстких суднових пластин. Пояснено незвичайні форми коливань квадратних пластин. Отримано фундаментальні й асимптотичні розв'язки бігармонічного рівняння вільних коливань жорстких суднових пластин.
Ключові слова: прямокутні пластини; форми вільних коливань; власні частоти коливань; бігармонічне рівняння; асимптотичний розв'язок.

Аннотация. Рассмотрены особенности расчета собственных частот и форм свободных колебаний прямоугольных жестких судовых пластин. Пояснены необычные формы свободных колебаний квадратных пластин. Получены фундаментальные и асимптотические решения бигармонического уравнения свободных колебаний судовых пластин.
Ключевые слова: прямоугольные пластины; формы свободных колебаний; собственные частоты колебаний; бигармоническое уравнение; асимптотическое решение.

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PROBLEM STATEMENT

The free-vibration problem of the rectangular plates is a classical subject for the investigation in different fields in mechanical, civil, aerospace and marine engineering [1–3, 5]. Traditionally the method of principal coordinates was the main instrumentation in the free vibration analysis of rectangular plates [3]. The majority of the theoretical and practical free-vibration and exited vibration problems have been solved by using this simple and transparent technique. New powerful impulse in the vibration analysis of the different engineering structures was generated by the using of numerical methods and computer codes based on FEM analysis of the structures. And now engineer, designer or investigator may practically use a lot of numerical instrumentation from the library of commercial computer packages like SolidWorks Simulation, ANSYS, Abaqus, MSC Nastran, etc. Nevertheless, in the field of classical analytical investigations of the vibration problems some problems are not solved exactly yet now. For example, an exact solution for a fully clamped rectangular plate has not been obtained yet now, and it is currently outlined in the publications [4, 7, 8] that an exact solution would not be achieved in the nearest future.

ANALYSIS OF THE LATEST STUDIES AND PUBLICATION

Hence to obtain correct mode shapes and natural frequencies of fully clamped rectangular plates a number of approximate methods were developed during the last century [1, 5]. These methods are based on different concepts like Fourier analysis and superposition principle, perturbation technique and asymptotic expansions, fundamental energy principles and algebraic procedures (Rayleigh–Ritz or Bubnov–Galerkin methods) etc. [1, 2, 5].

For example, the superposition of hybrid cylindrical Bessel functions for free vibration analysis of fully clamped rectangular plates is presented in modern paper [4]. Different approximate asymptotic methods observed in handbook [1]. A number of approximate methods for the calculation of free vibration mode shapes and natural frequencies ware especially discussed in report [5]. Most of the methods discussed ware based on the energy formulations of the problem and different numerical procedures.

Classical solutions for rectangular plates with typical side supports are presented in handbooks [1, 2]. Among the approximate solutions for the fully clamped plates to be presented in [1, 2], one of them is Iguchi method. This method includes polynomial and trigonometric approximations of the mode shapes and Fourier analysis of the resulting expressions. The method displays good results for the natural frequencies in comparison to some other methods [1, 2] and for mode shapes in comparison to FEM numerical results.

In addition to the above discussed aspects of the problem the application of the FEM technology displayed some unusual mode shapes for square plates both for hinge and clamped supports. It is important to say that in latest publications [6, 8] results included such unusual mode shapes among others, but no any attention has been paid by the authors about this phenomenon.

THE AIM OF THE ARTICLE. As the result of the discussion three goals have to be achieved in the article. The first goal is to revise the fundamental solutions of biharmonic equation and find out the exact solution for fully clamped rectangular plates. The second goal is to explain the existence of unusual mode shapes in the family of mode shapes for rectangular plates. And the third goal is to obtain simple and efficient asymptotic solutions for fully clamped plates.

PRESENTATION OF THE BASIC MATERIAL

1. Formulation and analytical solutions of the problem

Governing equation in the free vibration problem for the rigid rectangular plates is the biharmonic partial differential equation in Cartesian coordinates [3]

$$D\nabla^2\nabla^2 w(x, y) - \omega^2\rho h w(x, y) = 0\,, \qquad (1)$$

where $w(x, y)$ is the mode shape; operator $\nabla^2\nabla^2 = \partial^4/\partial x^4 + 2\partial^4/\partial x^2\partial y^2 + \partial^4/\partial y^4$ is biharmonic differential operator; ω is a natural frequency that is matched to the vibration mode $w(x, y)$; ρ is a mass density and h is a thickness of the plate (Fig. 1); $D = Eh^3/12(1-\nu^2)$ is the cylindrical bending rigidity of the plate; E and ν are being the Young's modulus and the Poisson's ratio coefficient, respectively.

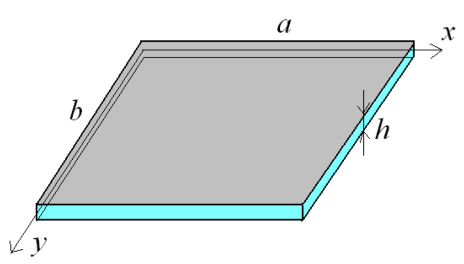


Fig. 1. A sketch of rectangular plate

For the complete formulation of the problem boundary conditions on the plate sides must be added to the Eq. (1). Because of only hinge and clamped (build in) side supports would be the interest of the analysis in the paper we should consider now formulation of the boundary conditions for these kinds of supports only. In particular, if some side of the plate is hinge constrained on the nonelastic support then boundary conditions are as follows: deflections and bending moments on the side must be equal zero: $w = 0$, $M_1 = 0$ or $M_2 = 0$, where $M_1 = -D\partial^2 w/\partial x^2$ stands for x-direction and $M_2 = -D\partial^2 w/\partial y^2$ stands for y-direction. Correspondently for clamped nonelastic side supports boundary conditions are: deflections and angles of rotation on the side must be equal to zero: $w = 0$, $\alpha_1 = 0$ or $\alpha_2 = 0$, where $\alpha_1 = \partial w/\partial x$ stands for x-direction and $\alpha_2 = \partial w/\partial y$ stands for y-direction. In the paper we should consider only fully hinge supported (simple supported) plates, fully clamped plates and plates with two opposite sides are simple supported but two other opposite sides are clamped.

For the analytical solution of the problem we should use Fourier (splitting) method. According to the method solution of the Eq. (1) is as follow

$$w(x, y) = X(x) \cdot Y(y) \neq 0 . \qquad (2)$$

Substitution of the solution (2) into the Eq. (1) transforms it to the form

$$\frac{X^{\mathrm{IV}}}{X} + 2\gamma^2 \frac{X^{\mathrm{II}}}{X}\frac{Y^{\mathrm{II}}}{Y} + \gamma^4 \frac{Y^{\mathrm{IV}}}{Y} - \tilde{\omega}^2 = 0, \qquad (3)$$

where $\gamma = a/b$ is a ratio coefficient for side dimensions of the plate (see Fig. 1); $\tilde{\omega}^2 = \omega^2 a^4 \rho h / D$ is nondimentinal square of natural frequency; $X^{\mathrm{II}} \equiv d^2X/d\xi^2$, $Y^{\mathrm{II}} \equiv d^2Y/d\eta^2$ and so on; normalized coordinates $\xi = x/a$; $\eta = y/b$ $(0 \leq (\xi, \eta) \leq 1)$ are used in Eq. (3).

Equation (3) wouldn't be spitted into two separate ordinary differential equations for unknown functions $X(x)$, $Y(y)$ and to do this we have to formulate some additional conditions for the functions $X(x)$, $Y(y)$. Equation (3) may be spitted into separate ordinary differential equations if: 1) $X^{\mathrm{II}}/X = C_1 = \mathrm{const}$ or 2) $Y^{\mathrm{II}}/Y = C_2 = \mathrm{const}$. Consider these two options in details.

In the first case when condition $X^{\mathrm{II}}/X = C_1 = \mathrm{const}$ takes place equation for $Y(y)$ would be as it follows $\gamma^4 Y^{\mathrm{IV}} + 2\gamma^2 C_1 Y^{\mathrm{II}} + (C_1^2 - \tilde{\omega}^2)Y = 0$. Solutions for the functions $X(x)$ and $Y(y)$ may be derived in the Eulerian form $X(x) = A\exp(\alpha x)$ and $Y(y) = B\exp(\beta y)$ with the following characteristic equations: $\alpha^2 - C_1 = 0$, $\gamma^4\beta^4 + 2\gamma^2 C_1 \beta^2 + (C_1^2 \times \tilde{\omega}^2)\beta = 0$ and their roots $\alpha_{1,2} = \pm\sqrt{C_1}$, $\beta_{1,2} = \pm\gamma^{-1}\beta_+$, $\beta_{3,4} = \pm\gamma^{-1}\beta_-$, where $\beta_\pm = \sqrt{C_1 \pm \tilde{\omega}}$. Solution for the function $X(x)$ would be in trigonometric functions if constant C_1 is a negative value say $C_1 = -(\sigma_1^-)^2$, and it would be in hyperbolic functions if C_1 is a positive value say $C_1 = (\sigma_1^+)^2$. Then the resulting expression for the first fundamental solution of biharmonic equation (1) would be

$$\begin{gathered} w_1(\xi, \eta) = X_1(\xi) \cdot Y_1(\eta) = \left(A_{1c}^+ \cosh \sigma_1^+ \xi + A_{1s}^+ \sinh \sigma_1^+ \xi\right) \times \\ \times \left[\left(B_{1c}^+ \cosh \beta_-^+ \eta + B_{1c}^+ \sinh \beta_-^+ \eta\right) + \left(b_{1c}^+ \cos \beta_+^+ \eta + b_{1c}^+ \sin \beta_+^+ \eta\right)\right] + \\ + \left(a_{1c}^- \cos \sigma_1^- \xi + a_{1s}^- \sin \sigma_1^- \xi\right) \times \\ \times \left[\left(B_{1c}^- \cosh \beta_+^- \eta + B_{1s}^- \sinh \beta_+^- \eta\right) + \left(b_{1c}^- \cos \beta_-^- \eta + b_{1s}^- \sin \beta_-^- \eta\right)\right], \end{gathered} \qquad (4)$$

where values $A_{1c,s}^\pm$, $B_{1c,s}^\pm$, $a_{1c,s}^\pm$, $b_{1c,s}^\pm$ are integration constants; $\sigma_1^\pm \geq 0$ are splitting constants and it is denoted $\beta_\pm^+ = \sqrt{\tilde{\omega} \pm (\sigma_1^+)^2}$, $\beta_\pm^- = \sqrt{\tilde{\omega} \pm (\sigma_1^-)^2}$.

In the second case when condition $Y^{\mathrm{II}}/Y = C_2 = \pm(\kappa_2^\pm)^2 = \mathrm{const}$ takes place solution of the problem may be derived in the same way and we would have

$$\begin{gathered} w_2(\xi, \eta) = X_2(\xi) \cdot Y_2(\eta) = \left(B_{2c}^+ \cosh \kappa_2^+ \eta + B_{2s}^+ \sinh \kappa_2^+ \eta\right) \times \\ \times \left[\left(A_{2c}^+ \cosh \alpha_-^+ \xi + A_{2s}^+ \sinh \alpha_-^+ \xi\right) + \left(a_{2c}^+ \cos \alpha_+^+ \xi + a_{2s}^+ \sin \alpha_+^+ \xi\right)\right] + \\ + \left(b_{2c}^- \cos \kappa_2^- \eta + b_{2s}^- \sin \kappa_2^- \eta\right) \times \\ \times \left[\left(A_{2c}^- \cosh \alpha_+^- \xi + A_{2s}^- \sinh \alpha_+^- \xi\right) + \left(a_{2c}^- \cos \alpha_+^- \xi + a_{2s}^- \sin \alpha_+^- \xi\right)\right], \end{gathered} \qquad (5)$$

where again values $A_{2c,s}^\pm$, $B_{2c,s}^\pm$, $a_{2c,s}^\pm$, $b_{2c,s}^\pm$ are integration constants; $\kappa_2^\pm \geq 0$ are splitting constants and it is denoted $\alpha_\pm^+ = \sqrt{\tilde{\omega} \pm (\kappa_2^+)^2}$, $\alpha_\pm^- = \sqrt{\tilde{\omega} \pm (\kappa_2^-)^2}$.

In the complete family of the fundamental solutions (4) and (5) $w(x, y) = w_1 + w_2$ we would consider partial solutions w_j, $j = 1, 2$ as independent solutions so that equality $a_1 w_1 + a_2 w_2 = 0$ takes place when all factors a_j are equal to zero.

2. Application of the fundamental solution to the rectangular plates

Consider now the application of the fundamental solution to the rectangular plates with typical side supports.

2.1. Fully simply supported plates. In this simplest case of the plate supports boundary conditions include only even orders of the derivatives in both directions as it takes place in the governing biharmonic equation (1) and there are no any problems in the satisfying of the boundary conditions for this kind of side supports. In particular, from the boundary conditions on the sides $\xi = 0$ and $\xi = 1$ of the plate it follows that; 1) $A_{1c,s}^+ = A_{2c,s}^+ = A_{1c,s}^- = A_{2c,s}^- = a_{1,2c}^+ = a_{1,2c}^- = 0$; 2) splitting constant σ_1^- and parameter α_-^- must be determined as the roots of the equations $\sin\sigma_-^- = 0$, $\sin\alpha_-^- = 0$. It means that $\sigma_1^- = m\pi$ and $\alpha_-^- = \sqrt{\tilde{\omega} - (\kappa_2^-)^2} = m_1\pi$, m, $m_1 = 1, 2, 3, \ldots$ correspondently. The same boundary conditions on the sides $\eta = 0$ and $\eta = 1$ would give analogical results for the integration constants $B_{1,2c,s}^\pm$, $b_{1,2c}^\pm$ and conditions $\sin\kappa_2^- = 0$, $\sin\beta_-^- = 0$

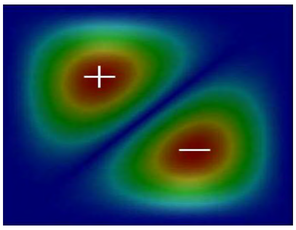
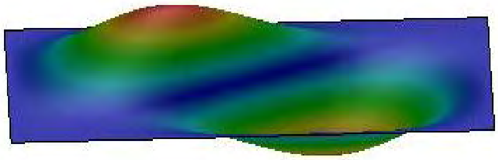
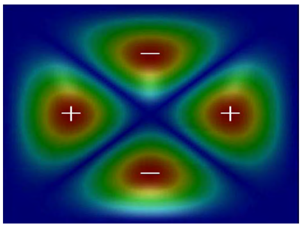
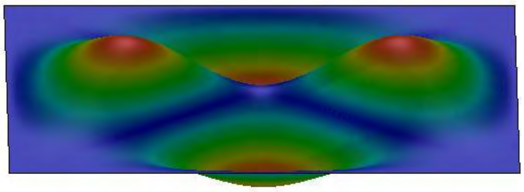
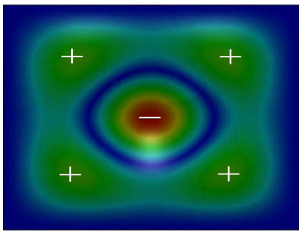
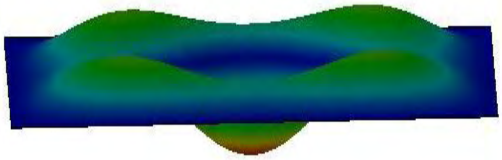
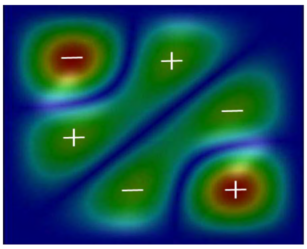
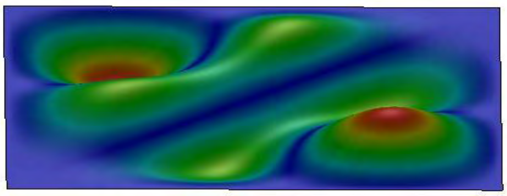
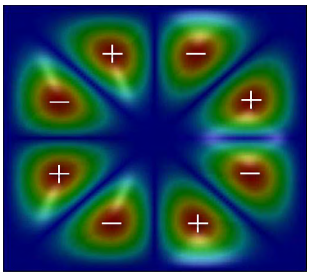
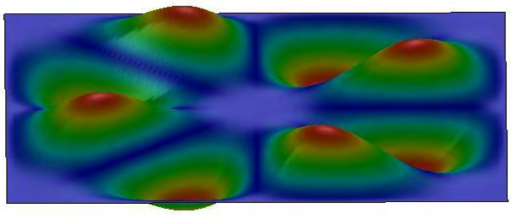
so we can write again that $\kappa_2^- = n\pi$ and $\beta_-^- = \sqrt{\tilde{\omega} - (\sigma_1^-)^2} = n_1\pi$, $n, n_1 = 1, 2, 3, \ldots$. Finally solution for the mode shapes and natural frequencies would be

$$w_{mn}(x, y) = a_{1s}^- b_{1s}^- \sin m\pi\xi \cdot \sin n\pi\eta + a_{2s}^- b_{2s}^- \sin m_1\pi\xi \cdot \sin n_1\pi\eta,$$
$$\tilde{\omega}_m = \pi^2 (m^2 + \gamma^2 n^2) = \pi^2 (m_1^2 + \gamma^2 n_1^2), m, n, m_1, n_1 = 1, 2, 3, \ldots . \quad (6)$$

In particular case when $m = m_1$ and $n = n_1$ we have got the well-known classical result [1, 2], but solution (6) has a little more general form and the question is what does it mean?

Application of the FEM technology for the solution of the problem displayed some specific mode shapes that ware looking very unusually in comparison to the traditional mode shapes for rectangular plates [1, 2] (numerical calculations were carried out by Master of engineering D. Litvinenko). In Table 1 some unusual mode shapes are illustrated.

Table 1. **Unusual mode shapes of square plate**

№	Unusual mode shapes	
	Top view	Side view
1		
2		
3		
4		
5		

From the classical point of view positive and negative deflections of the mode shapes must be positioned periodically in both directions over the plate, but in the Table 1 one can see specific skewed, axially symmetric and daisy forms of the mode shapes. Classical and modern publications [1, 4–8] don't concentrate attention on these unusual mode shapes and we think that this phenomenon may be explained it the framework of solution (6).

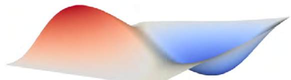
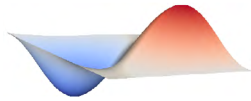
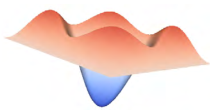
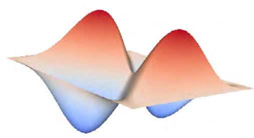
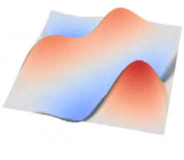
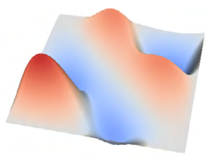
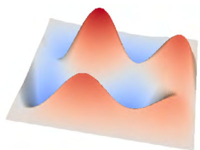
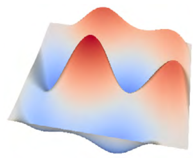
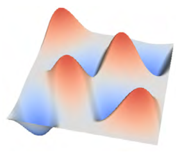
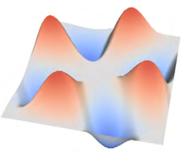
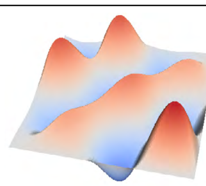
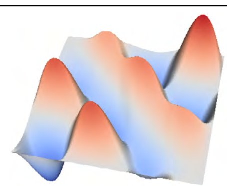
From the formula for the natural frequencies in Eq. (6) it follows that mode shapes with different numbers (m, n) and (m_1, n_1) would refer to the same natural frequency when condition $(m^2+\gamma^2 n^2) = (m_1^2+\gamma^2 n_1^2)$ takes place. For the simple case of square plates when $\gamma = 1$ it means that equalities $m_1 = n$, $n_1 = m$ take place. In more general case of rectangular plates with the rational value of the ratio coefficient $\gamma = k/s$, $k, s = 1, 2, 3\ldots$ corresponding equalities are $m_1 = nk/s$, $n_1 = ms/k$ within the condition $m_1 n_1 = nm$. Some combinations of m_1 and n_1 are presented in the Table 2.

Table 2. **Some combinations of the mode numbers for unusual mode shapes**

$\gamma = k/s$	$m\times n = m_1\times n_1$			
1/2	2×2 = 1×4	3×4 = 2×6	4×4 = 2×8	….
2/3	4×3 = 2×6	6×3 = 2×9	6×6 = 4×9	….
3/4	6×4 = 3×8	9×4 = 3×12	9×8 = 6×12	….
3/5	6×5 = 3×10	9×5 = 3×15	9×10 = 6×15	….

Hence when different mode shapes correspond to the same value of natural frequency we have to summarize these mode shapes in the form (6). Samples are illustrated in Table 3. This phenomenon may be named as superposition effect for unusual mode shapes.

Table 3. **Samples of unusual mode shapes**

Superposition	Side view	Superposition	Side view
$w_{12} = X_1Y_2 + X_2Y_1$		$w_{12} = X_1Y_2 - X_2Y_1$	
$w_{13} = X_1Y_3 + X_3Y_1$		$w_{13} = X_1Y_3 - X_3Y_1$	
$w_{23} = X_2Y_3 + X_3Y_2$		$w_{23} = X_2Y_3 - X_3Y_2$	
$w_{14} = X_1Y_4 + X_4Y_1$		$w_{14} = X_1Y_4 - X_4Y_1$	
$w_{24} = X_2Y_4 + X_4Y_2$		$w_{24} = X_2Y_4 - X_4Y_2$	
$w_{34} = X_3Y_4 + X_4Y_3$		$w_{34} = X_3Y_4 - X_4Y_3$	

One can see that superposition of the mode shapes for w_{12}, w_{23}, w_{14}, and w_{34} have an anti symmetric structure, but superposition of the mode shapes for w_{13} and w_{24} have an opposite structure.

2.2. Plates with combined supports for opposite sides. Consider the case when sides $\xi = 0$ and $\xi = 1$ of the plate are clamped but other two opposite sides $\eta = 0$ and $\eta = 1$ are simply supported. Then integration constants in the y-direction must be the same as in 2.1, but boundary conditions $w = \partial w/\partial x = 0$ on the sides in the x-direction generate the following expressions

$$\left.\begin{aligned} &1)\ a_{1c,s}^{-} = 0,\ a_{2c}^{-} = -A_{2c}^{-},\ a_{2s}^{-} = -\nu_\alpha A_{2s}^{-},\ A_{2c,s}^{-} \neq 0;\ \nu_\alpha = \alpha_+^{-}/\alpha_-^{-},\\ &2)\ 2\nu_\alpha\left(1-\cosh\alpha_+^{-}\cos\alpha_-^{-}\right)-\left(1-\nu_\alpha^{\,2}\right)\sinh\alpha_+^{-}\sin\alpha_-^{-} = 0.\end{aligned}\right\} \quad (7)$$

Second expression in Eq. (7), to be considered as the condition that $w(x, y) \neq 0$, after some algebraic manipulations can be transformed to the well-known form [3] $\left(\tanh u_+ - \nu_\alpha \tan u_-\right)\left(\tanh^{-1} u_+ - \nu_\alpha \tan^{-1} u_-\right) = 0,\quad u_\pm = \alpha_\pm^{-}/2.$

Finally expression for the mode shapes is as follow

$$w_n(x,y) = A_{2c}^{-}\left[\left(\cosh\alpha_{+}^{-}\xi - \cos\alpha_{-}^{-}\xi\right) - \delta_\alpha\left(\sinh\alpha_{+}^{-}\xi - \nu_\alpha \sin\alpha_{-}^{-}\xi\right)\right]\sin n\pi\eta, \quad (8)$$

where $\delta_\alpha = (\sinh\alpha_{+}^{-} - \nu_\alpha \sin\alpha_{-}^{-})/(\cosh\alpha_{+}^{-} - \cos\alpha_{-}^{-})$ and values $\alpha_{\pm}^{-} = \sqrt{\tilde{\omega} \pm \gamma^2 (n\pi)^2}$, $n = 1, 2, 3, \ldots$ must be derived as the roots $(\alpha_{\pm}^{-})_{mn}$ of the transcendental Eq. (7). Correspondingly the resulting expression for the natural frequencies would be $\tilde{\omega}_{mn} = (\alpha_{+}^{-})_{mn}^2 - \gamma^2(n\pi)^2 = (\alpha_{-}^{-})_{m_1 n_1}^2 + \gamma^2(n\pi)^2$.

2.3. Fully clamped plate. Fully clamped plates are the most difficult case from the point of view of analytical solution of the free-vibration problem. In this case odd order of the derivatives in the boundary conditions in both directions doesn't match with the even order of the derivatives in the governing equation and this fact generates the main difficulties in the analytical solution of the problem. Substitution of the fundamental solution (3) and (4) into the boundary conditions $w = \partial w/\partial x = 0$ on the sides $\xi = 0$ and $\xi = 1$ and $w = \partial w/\partial y = 0$ on the sides $\eta = 0$ and $\eta = 1$ of the plate leads to the final results that all the integration constants in the solution for this kind of supports torn to zero. The main reason for this related with the even derivatives in governing equation but odd derivatives in boundary conditions.

3. Asymptotic solution for fully clamped rectangular plates

Consider in this chapter approximate asymptotic solutions for fully clamped plates. Investigators have found quite a long time ago that mode shapes in the majority part of the plate are the mode shapes of simply supported plates, but with some minor transformations close to the plate sides [5] (Fig. 2).

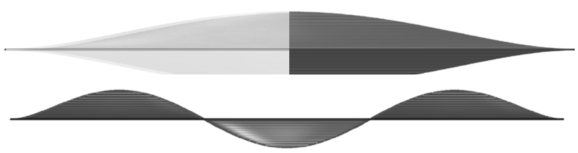


Fig. 2. Samples of the mode shape sections for fully clamped plate

Different asymptotic techniques are based on these findings. Consider for the beginning so-called matching asymptotic approach. According to this approach plate area separates into the *outer* and *inner* domains (Fig. 3).

For inner solution of the problem mode shapes may be written in the form: $w_{mn}^{\text{in}}(x,y) = \sin[m\pi(x-\Delta a)/a_e] \times \sin[n\pi(y-\Delta b)/b_e]$, where values $a_e = a - 2\Delta a$ and $b_e = = b - 2\Delta b$ are the effective (reduced) dimensions of the plate.

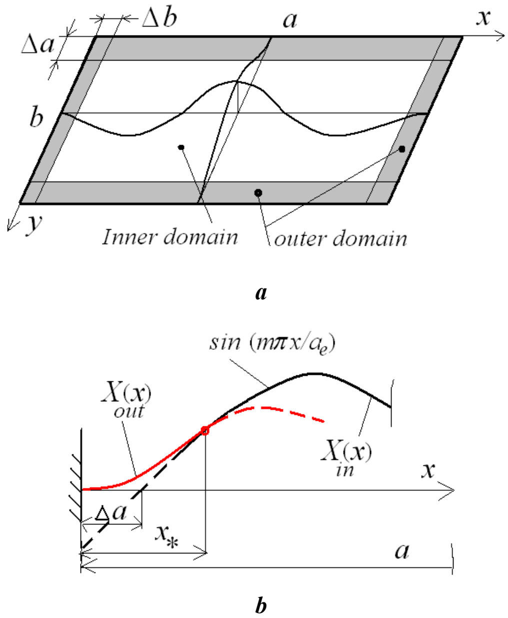


Fig. 3. Sketch of the inner and outer domains for the rectangular plate (***a***), and matching of the outer and inner solutions for $X(x)$ (***b***)

Because of the outer domain is the collection of very narrow strips along plate the outer solution in x-direction in the basic approach would be $w_{mn}^{\text{out}}(x,y) = X^{\text{out}}(x)\cdot Y_n^{\text{in}}(y) = \sum_{i=0}^{3} A_i x^i \cdot \sin[n\pi(y-\Delta b)/b_e]$. Analogically in y-direction $w_{mn}^{\text{out}}(x,y) = \sum_{i=0}^{3} B_i y^i \cdot \sin[m\pi(x-\Delta a)/a_e]$. From the boundary conditions on the clamped sides of the plate it follows that $A_0 = A_1 = 0$ and $B_0 = B_1 = 0$. To determine the other coefficients we should use matching condition in some points $x = x_*$ and $y = y_*$ for the deflections and first derivatives: $w_{mn}^{\text{out}} = w_{mn}^{\text{in}}$, $\partial w_{mn}^{\text{out}}/\partial x = \partial w_{mn}^{\text{in}}/\partial x$ and $\partial w_{mn}^{\text{out}}/\partial y = \partial w_{mn}^{\text{in}}/\partial y$ correspondingly.

Additionally for the determination of the unknown matching points x_*, y_* and reduction values Δa and Δb for the plate sides we should us matching conditions for the second and third derivatives of the both solutions. After all mathematical manipulations results would be: $x_* = 2.45\Delta a$ within the condition $m\pi x_*/a_e \cong 1$ from which it follows that $\Delta a \equiv \Delta a_m = a/(2+2.4m\pi)$, $m = 1, 2, 3, \ldots$. And finally the outer part of the solution in x-direction is as follows

$$w_n^{\text{out}}(x,y) = \left[0.623\xi_*^2(1-0.222\xi_*)\right]\cdot \sin[n\pi(y-\Delta b)/b_e],$$

$$\xi_* = x/x_* \in [0;1], \quad (9)$$

with the equation for the spectrum of natural frequencies $\tilde{\omega}_{mn} = \pi^2[(m+\nu_m)^2 + \gamma^2(n+\nu_n)^2]$, $m, n = 1, 2, 3, \ldots$, where $\nu_m = \nu_n = 1/1.2\pi = 0.267$.

Comparison the results of the matching asymptotic solution (9) for mode shapes with the numerical results obtained in the framework of FEM technology displayed quite good coincidence, but comparison the results for natural frequencies with the Iguchi calculations [1] is not acceptable. Value $\nu_m = 0.267$ is off the interval [0.34; 0.48] for Iguchi approximate solution.

Then consider another version of the solution (boundary perturbated solution) which is based on the assumption that perturbations generated by the boundary conditions diminish very fest and practically are zero in the inner domain of the plate. In that case mode shapes may be written in the form [1]

$$w_{mn}(x,y) \cong \left[\sin m\pi(x-\Delta a_{mn})/a_e + \Delta X_{mn}(x)\right]\times \\ \times\left[\sin n\pi(y-\Delta b_{mn})/b_e + \Delta Y_{mn}(y)\right], \qquad (10)$$

where functions $\Delta X_{mn}(x)$ and $\Delta Y_{mn}(y)$ must torn to zero in the inner domain of the plate and $a_e = a - 2\Delta a_{mn}$, $b_e = b - 2\Delta b_{mn}$ are reduced dimensions of the plate.

Substitution of the solution (10) into the governing equation and boundary conditions lids finally to the following results:

1. Equation for natural frequencies:

$$\tilde{\omega}_{mn} = \pi^2\left[(m+\nu_{mn})^2 + \gamma^2(n+\nu_{nm})^2\right], \qquad (11)$$

where $\nu_{mn} = 2\pi\Delta a_{mn}/a_e$, $\nu_{nm} = 2\pi\Delta b_{nm}/b_e$.

2. Boundary perturbated parts of the mode shapes:

$$\left.\begin{aligned}\Delta X_{mn}(\xi) &= a^-_{mn}\exp(-r_{mn}\xi) + (-1)^{m+1}a^-_{mn}\exp(-r_{mn}(1-\xi)),\\ \Delta Y_{mn}(\eta) &= b^-_{mn}\exp(-\bar{r}_{mn}\eta) + (-1)^{n+1}b^-_{mn}\exp(-\bar{r}_{mn}(1-\eta)),\end{aligned}\right\} \qquad (12)$$

where it is denoted $a^-_{mn} = \sin(-q_m\Delta a_{mn}/a)$, $b^-_{mn} = \sin(-p_n\Delta b_{nm}/b)$; $r_{mn} = \sqrt{q_m^2 + 2\gamma^2 p_n^2}$, $\bar{r}_{mn} = \sqrt{q_m^2 + 2\gamma^{-2} p_n^2}$; $p_n = n\pi b/b_e$ $q_m = m\pi a/a_e$ and factors $(-1)^{n+1}$ and $(-1)^{m+1}$ regulates even and odd mode shapes.

3. Equations for the reduction values Δa_{mn} and Δb_{nm}:

$$\left.\begin{aligned}\tan(m\pi\Delta a_{mn}/a_e) &= \frac{1}{\sqrt{1+2\gamma^2\, p_n^2/q_m^2}},\\ \tan(n\pi\Delta b_{mn}/b_e) &= \frac{1}{\sqrt{1+2\gamma^{-2}\, q_m^2/p_n^2}}.\end{aligned}\right\} \qquad (13)$$

Because the algorithm (10)–(13) have recurrent structure in Eq. (13), iteration procedure must be applied in the calculations of the values Δa_{mn} and Δb_{nm}. But systematic numerical calculation displayed very minor corrections in the results for natural frequencies and in general no more than one iteration procedure may be recommended in the practical calculations.

Comparison of the author's results for natural frequencies ω_{mn} with approximate Iguchi results for the values m, n = 1, 3, 5 and γ = 1.0 and 0.5 (see [1]) presented in Table 4. Values Δ in the table present discrepancies (%) between these results for natural frequencies.

Table 4. **Comparison of the author results for natural frequencies ω_{mn} (lower values in the rows) with Iguchi results (upper values)**

	γ	m = 1		3		5	
n = 1	1.0	36.0	**Δ, %**	131.9	**Δ, %**	309.0	**Δ, %**
		35.1	**2.5**	131.6	**0.23**	308.9	**0.03**
	0.5	24.6	**1.6**	124.0	**0.65**	302.0	**0.36**
		24.2		123.2		300.9	
3	1,0	131.9	**0,23**	220.1	**0.36**	393.4	**0.20**
		131.6		219/3		392.8	
	0,5	44.8	**0,89**	142.4	**0.21**	320.1	**0.3**
		44.4		142.1		320.0	
5	1.0	309.0	**0.03**	393.4	**0.20**	562.2	**0.12**
		308.9		392.8		561.5	
	0.5	87.3	**0.34**	181.8	**0.22**	358.0	**0.22**
		87.0		181.4		358.8	

Only for two frequencies ω_{11} and ω_{12} the values of the discrepancies are large then 1 % and are localized in the interval [1.4…2.5 %], but for all other frequencies ω_{mn}, $m, n \geq 2$ values of the discrepancies are smaller and much smaller than 1 %. It means that proposed algorithm may be used as vary simple and no labor procedure for the estimation of natural frequencies for fully clamped rectangular plates.

Comparison of some mode shapes derived by using FEM technology and with asymptotic solutions elaborated in the paper is illustrated in Fig. 4.

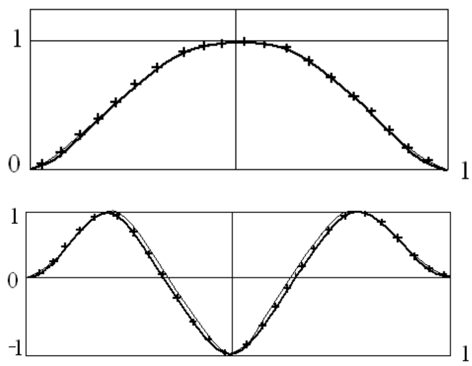


Fig. 4. Results for the first and third mode shapes of fully clamped plate: ----- – FEM: —— – matching solution; + + + – boundary perturbated solution

CONCLUSIONS

1. Application of the fundamental solution to the fully clamped plates displayed that the solution can't satisfy boundary conditions in that case (all constants of the integration torn to zero), but for the other side supports the results of the application ware successive. The main reason for this failure is that biharmonic equation includes only even derivatives but boundary conditions include odd derivatives.

2. Two versions of asymptotic solutions for the fully clamped plates displayed good results for mode shapes and second version displayed good results for natural frequencies too.

3. Unusual mode shapes (skewed, axially symmetric and daisy shapes) take place when different vibration modes have the same natural frequencies $\omega_{mn} = \omega_{m1n1}$. In these special situations unusual mode shapes are the result of the superposition of the partial mode shapes $(X_m Y_n \pm X_{m1} Y_{n1})$.

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