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GENERALIZED MULTI-SCALE TECHNIQUE FOR THE SOLUTION OF HIGH ORDER NONLINEAR SHRÖDINGER EQUATIONS FOR DEEP-WATER SURFACE WAVES

A. N. Serdyuchenko, professor, doctor of phys. and math. sciences
T. V. Emeljanova, lecturer

National University of Shipbuilding, Mykolayiv

Abstract. The generalized multi-scale technique for the solution of Nonlinear Schrödinger Equations (NLS) of the 3-rd to 5-th order in nonlinearity for surface deep-water waves has been developed. The generalization is directed to suppressing the nonphysical secular solutions in the high approaches. The complete family of the stationary solutions of the 3-rd order NLS equation has been derived in terms of elliptical Vejerstrass and Jacobi functions.

Keywords: surface water waves, nonlinearity, Schrödinger equations, stationary solutions.

Анотація. Розроблено узагальнений метод багатьох масштабів для розв'язання нелінійних рівнянь Шредінгера (НРШ) 3–5 порядків за нелінійністю для поверхневих хвиль на глибокій воді. Узагальнення спрямовані на подолання нефізичних секулярних розв'язків у старших наближеннях. Отримано повний клас стаціонарних розв'язків для НРШ третього порядку у термінах еліптичних функцій Веєрштрасса і Якобі.

Ключові слова: поверхневі хвилі на воді, нелінійність, рівняння Шредінгера. стаціонарні розв'язки.

Аннотация. Разработан обобщенный метод многих масштабов для решения нелинейных уравнений Шредингера (НУШ) 3–5 порядков по нелинейности для поверхностных волн на глубокой воде. Обобщения предприняты с целью подавления нефизических секулярних решений в старших приближениях. Получен полный класс стационарных решений для НРШ третьего порядка в терминах эллиптических функций Веерштрасса и Якоби.

Ключевые слова: поверхностные волны на воде, нелинейность, уравнения Шредингера, стационарные решения.

INTRODUCTION

In the beginning of the second part of the last century after the discovery of the famous Benjamin and Fair amplitudes envelope instability of weakly nonlinear surface water waves, the Nonlinear Schrödinger Equations of different orders became the most effective and powerful theoretical technique for the investigations of modulated wave trains [3-5]. The first attempts of using the technique were related with the solutions of the 3-rd order NLS equations — the lowest order of the equations for deep-water surface waves [3, 9, 13, 18]. A number of soliton and periodic solutions of the equations have been derived for the formulations of different physical problems [3, 4, 9, 13, 16, 18]. Experimental investigations have proved the effectiveness of the NLS technique [6, 7, 11].

Then NLS equations of the 4-th and 5-th orders in the nonlinearity were involved into the consideration [5, 8, 10, 11, 14, 15]. These equations included the effects of drift mass transport in nonlinear waves groups and higher order nonlinear and dispersion effects in wave motions, and allowed to describe the recurrence and frequency down-shift phenomena [7, 17]. But the attempts to obtain the asymptotical solutions of the NLS equations of high order displayed the existence of nonphysical secular terms in the solutions [3, 15].

THE MAIN GOALS OF THE PAPER are as follows: 1) to develop the generalized version of the asymptotical solution of high order NLS equations by using the multi-scale technique for eliminating secular terms from the solutions; 2) to derive the most complete family of the generalized stationary solutions of the 3-rd order NLS equation.

1. NLS EQUATIONS FOR AMPLITUDE ENVELOPE OF DEEP WATER WAVES

First of all, we briefly describe the procedure of derivation of NLS equations for deep-water surface waves. The original formulation of the boundary-value problem for deep water surface waves in the irrigational and potentional approximation of the wave motion is as follows [5, 15, 18]:

$$\left.\begin{aligned}&\nabla^2\Phi_w = 0, \quad z<\zeta_w;\\ &\zeta_{wt} + \left(\nabla\Phi_w\cdot\nabla(\zeta_w - z)\right) = 0, \quad z=\zeta_w;\\ &\Phi_{wt} + \frac{1}{2}\left(\nabla\Phi_w\cdot\nabla\Phi_w\right) + gz = -p^a/\rho, \quad z=\zeta_w;\\ &\nabla\Phi_w \to 0, \quad z\to-\infty; \quad \nabla\Phi_w<\infty, \left(x^2+y^2\right)^{1/2}\to\infty\end{aligned}\right\}, \tag{1}$$

where $\Phi_w(\vec{x},t)$ is velocity potential, $z=\xi_w(x,y,t)$ is wave surface, $p^a(x,y,t)$ is surface wind pressure and $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ is the Hamiltonian operator.

The formulation (1) is nonlinear and the main difficulties in the solution of the problem are related with the nonlinear boundary conditions on the unknown wave surface. For weakly nonlinear approach, it is possible to simplify the formulation (1) by using the Taylor series for the potential function $\Phi_w(\vec{x},t)$ in the vicinity of nondisturbed still-water plane. Then the nonlinear boundary conditions on the wave surface in (1) may be resulted in the following modified form on the plane $z=0$ [14, 15]:

$$\sum_m\sum_n\left(\frac{-1}{g}\right)^m\frac{1}{m!}\frac{1}{n!}\left\{Z_0^n(D+\widetilde{p})^m\left[L+d_t(D+\widetilde{p})\right]\right\}_z^{(m+n)}=0, \quad z=0, \tag{2}$$

where $m,n=0,1,2,\ldots$; $d_t = \left[\nu\partial/\partial t + \left(\nabla\Phi_w\cdot\nabla\right)\right]$ and factor $\nu=2$ for the derivative from the term D and $\nu=1$ for the derivative from the pressure $\widetilde{p}$ respectively. Eq. (2) includes the following operators

$$Z_0 = -\Phi_{wt}/g, \quad L(\Phi_w) = \left(\Phi_{wtt} + g\Phi_{wz}\right),$$

$$D(\Phi_w) = \left(\nabla\Phi_w\cdot\nabla\Phi_w\right)/2, \quad \widetilde{p} = p^a/\rho.$$

In addition to eq. (2) there is an expression for wave elevations in the explicitly form [14, 15]

$$\zeta_w = \sum_n\frac{1}{n!}\left(Z_w^n\right)_z^{(n-1)}, \quad Z_w = \frac{-1}{g}\left[\Phi_{wt} + \left(\nabla\Phi_w\cdot\nabla\Phi_w\right)/2 + p^a/\rho\right], \quad z=0.$$

To solve the modified boundary-value problem, the multi-scale technique has been used within the basic formulations

$$\left.\begin{aligned}&\Phi_w = \varepsilon\left(\Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \cdots\right), \quad \Phi_n = \Phi_n\left(\vec{X}_m, T\right);\\ &\vec{X} = \varepsilon\langle k\rangle\vec{x}, \quad T=\varepsilon\langle\sigma\rangle t, \quad |\varepsilon| \ll 1, \quad \vec{x} = (x,y,z),\end{aligned}\right\}, \tag{3}$$

where ε is a small parameter considered as a measure of nonlinearity and modulations effects in the wave motion, $\langle\sigma\rangle, \langle k\rangle = \langle\sigma\rangle^2/g$ are the average values of frequency and wave number in the weakly irregular wave field respectively and $\vec{X}, T$ are the slowly-varying special and temporal coordinates, respectively.

The application of the multi-scale technique [3, 6, 9, 14, 15, 18] to the nonlinear boundary-value problem (1), (2) allows us to transform it into the recurrent set of the linear boundary-value problems in the perturbations for the unknown potential functions Φ_n in the series (3). The solutions of these linear boundary-value problems up to the 5-th order including have been derived and the results are as follows

$$\left.\begin{aligned} &\Phi_0 = -i\tilde{A}\left(\vec{X},T\right)\exp\Theta + c.c., \quad \Phi_1 = \tilde{\Phi}_{dr}\left(\vec{X},T\right); \\ &\Phi_2 = 0, \quad \Phi_3 = -i\tfrac{1}{2}A_{32}\left(\vec{X},T\right)\exp 2\Theta + c.c., \\ &\Phi_4 = -i\tfrac{1}{2}A_{42}\left(\vec{X},T\right)\exp 2\Theta - i\tfrac{1}{3}A_{43}\left(\vec{X},T\right)\exp 3\Theta + c.c., \end{aligned}\right\}, \tag{4}$$

where $\Theta = \zeta + i\theta = \langle k\rangle z + i\left(\langle\sigma\rangle t + \langle k\rangle x\right)$ is the complex phase coordinate, $\tilde{\Phi}_{dr}$ is a drift potential function; the amplitudes of high bounded harmonics A_{jk} depend both on the complex amplitude of the fundamental wave mode

$$\tilde{A} = \tfrac{1}{2}a\exp i\psi$$

and on the amplitudes of high harmonics A_j^a, $j = 1,2,\ldots$ in surface wind pressure for which the order of magnitude is assumed to be about $p^a \sim O(\varepsilon^n)$, $n \geq 3$, (for the severe sea conditions).

The elimination of the secular terms from the solutions (4) generated the so called NLS equations for the amplitude $\tilde{A}\left(\vec{X},T\right)$ which we would consider in the framework of paper for the two dimensional free wave motion for the simplicity. The 5-th order NLS equation can be written in the operator form as follows

$$\begin{aligned} &\left(2A_T - A_X\right) + L(A) - i\delta_\varepsilon^2 A|A|^2\left(1 + \frac{5}{4}\varepsilon\delta_\varepsilon^2|A|^2\right) - \\ &-\frac{1}{2}\varepsilon\delta_\varepsilon^2\left[D_{1X}(A) - \frac{i}{2}\varepsilon D_{2X}(A) + 2iF(\Phi_{dr})\right] + O\left(\varepsilon^4\right) = 0, \quad z = 0, \end{aligned} \tag{5}$$

where the following notations for the operators have been used

$$\left.\begin{aligned} &L(A) = \left[-\frac{i}{4}\varepsilon A_{XX} + \frac{1}{8}\varepsilon^2 A_X^{(3)} + \frac{5i}{64}\varepsilon^3 A_X^{(4)}\right], \quad D_{1X}(A) = \left(7|A|^2 A_X - A|A|_X^2\right); \\ &D_{2X}(A) = \left[\left(D_{1X}\right)_X - |A|^2 A_{XX} + \frac{1}{2}A_X|A|_X^2 - \frac{23}{2}A|A|_X^2\right]; \\ &F(\Phi_{dr}) = \left[A\Phi_{drX} - i\varepsilon\left(\frac{1}{4}A\Phi_{drXX} + A_X\Phi_{drX}\right)\right] \end{aligned}\right\}. \tag{6}$$

Here in eq. (5) $\delta_\varepsilon^2 = \langle\delta\rangle^2/\varepsilon$, the following normalization of the variables has been used: $A = 2\varepsilon\tilde{A}$, $\Phi_{dr} = 2\varepsilon^2\langle\delta\rangle^{-2}\tilde{\Phi}_{dr}$, $\langle\delta\rangle = \langle k\rangle\langle a\rangle$.

In addition to eq. (5) and (6) the following linear boundary-value problem for the drift potential function $\tilde{\Phi}_{dr}$ results from the multi-scale technique

$$\left.\begin{aligned} &\Phi_{dr_{XX}} + \Phi_{dr_{ZZ}} = 0, \quad z < 0; \\ &\Phi_{dr_Z} + \varepsilon\Phi_{dr_{TT}} = |A|_X^2 - \frac{3}{4}i\varepsilon\left(|A|_{XX}^2 - 2\left(AA_X^*\right)_X\right) + O\left(\varepsilon^2\right), \quad z = 0; \\ &\Phi_{dr_Z} \to 0, \quad z \to -\infty; \qquad \Phi_{dr_X} \to 0, \quad |x| \to \infty \end{aligned}\right\}. \tag{7}$$

And finally in the fluid domain $z < 0$ amplitude A has to satisfy the linear equation

$$A_X - iA_Z = 0,$$
$$z < 0; \quad A = \tilde{A}, \quad z = 0, \tag{8}$$

where $Z = \varepsilon z$ is the slowly-varying vertical coordinate.

Equations (5)–(8) generate complete system of the governing equations in the perturbations.

2. GENERALIZED MULTI-SCALE TECHNIQUE FOR THE SOLUTION OF NLS EQUATIONS

Here in this section of the paper we would describe the main formulations of the generalized algorithm of the multi-scale technique which includes three steps. On the first step we would modify original slowly-varying independent variables (X, T) to the new coordinates $\xi = 2(X+T/2)$, $\tau = \varepsilon T/2$ so that the first new coordinate ξ is the variable in the coordinate system moving with the group velocity, and the second coordinate τ determines the evolution of the wave groups caused by dispersion and nonlinear affects of the wave motion. In the new variables the derivatives would be

$$\frac{\partial}{\partial T} = \frac{\partial}{\partial \xi} + \frac{1}{2}\varepsilon\frac{\partial}{\partial \tau},\ \frac{\partial^n}{\partial X^n} = 2^n\frac{\partial^n}{\partial \xi^n},\ n = 1, 2, \ldots$$

and NLS equation (6) transforms to the following operator form

$$N_L^{III} + i\varepsilon N_L^{IV} + \varepsilon^2 N_L^V + O\left(\varepsilon^3\right) = 0, \quad (9)$$

within the operators in the perturbations

$$\left.\begin{aligned}
&N_L^{III}(A) = \left(iA_\tau + A_{\xi\xi} + \delta_e^2 A|A|^2\right),\\
&N_L^{IV} = \left(N_L^{III}\right)_\xi - \Delta N_L^{IV},\\
&N_L^V = -\frac{5}{4}\left(N_L^{IV}\right)_\xi \Delta N_L^V = -\frac{5}{4}\left[\left(N_L^{III}\right)_{\xi\xi} - \left(\Delta N_L^{IV}\right)_\xi\right] + \Delta N_L^V
\end{aligned}\right\}, \quad (10)$$

where the «residual» parts of the high order operators N_L^{IV}, N_L^V are

$$\left.\begin{aligned}
&\Delta N_L^{IV} = \left[iA_{\xi\tau} + 8\delta_e^2|A|^2 A_\xi + 2i\delta_e^2 A\Phi_{dr_\xi}\right]\\
&\Delta N_L^V = \frac{5}{4}\delta_e^4 A|A|^4 + D_2(A) + F_2(\Phi_{dr})
\end{aligned}\right\}, \quad (11)$$

In eq. (10), (11) $\delta_e = \langle\delta\rangle/\varepsilon$ and operators $D_2(A)$ and $F_{2\xi}$ transform to the following form in the new variables

$$D_2(A) = \delta_e^2\left[\left(-14\frac{3}{4}|A|^2 A_{\xi\xi} - 14|A|_\xi^2 A_\xi + \frac{9}{4}|A|_{\xi\xi}^2 A\right) + \frac{23}{2}A\left|A_\xi\right|^2\right],$$

$$F_2(\Phi_{dr}) = -\frac{i}{2}\delta_e^2\left(7A\Phi_{dr\xi\xi} + 13A_\xi\Phi_{dr\xi}\right).$$

The operator $N_L^{III}(A)$ represents the basic NLS equation of the 3-rd order; addition to the basic equation the operator $N_L^{IV}(A)$ transforms it to the NLS equation of the 4-th order, and finally addition the 4-th order equation the operator $N_L^V(A)$ generates the NLS equation of the 5-th order.

The second step of the algorithm is related with the asymptotic expansions of the amplitudes envelope $a = a(\xi, \tau, \varepsilon)$ and perturbation phase $\psi = \psi(\xi, \tau, \varepsilon)$ in the complex amplitude $A = a \cdot \exp i\psi$ in the following form [12, 18]

$$\left.\begin{aligned}
a(\xi,\tau,\varepsilon) &= a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots,\\
\psi(\xi,\tau,\varepsilon) &= \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \ldots,
\end{aligned}\right\}, \quad (12)$$

where $a_j(\xi, \tau)$ and $\psi_j(\xi, \tau)$ are the unknown functions to be determined from the perturbation equations. Then any operator in the NLS equation (9) $N_L^\alpha = \left(N_R^\alpha + iN_J^\alpha\right)\exp i\psi$, where sub indexes «R» and «J» define real and imaginary parts of the operator, can be expressed in the following asymptotical series

$$N_{R,J}^\alpha = \left(\sum_j \varepsilon^j a_j, \sum_j \varepsilon^j \psi_j\right) = N_{R,J0}^\alpha(a_0,\psi_0) + \varepsilon N_{R,J1}^\alpha(a_1,\psi_1;a_0,\psi_0) + \\ + \varepsilon^2 N_{R,J2}^\alpha(a_2,\psi_2;a_0,\psi_0;a_1,\psi_1) + O\left(\varepsilon^3\right).$$

For example, the explicit expressions for the operators $N^{III}_{R,Jn}$, $n=0,1$ are as follows

$$\left.\begin{aligned}
N^{III}_{R0} &= \left(-a_0,\psi_{0_\tau} + a_{0_{\xi\xi}} - a_0\psi^2_{0\xi}\delta^2_e a^3_0\right),\\
N^{III}_{R1} &= \left[-\left(a_0,\psi_{1_\tau} + a_1,\psi_{0_\tau}\right) + a_{1_{\xi\xi}} - \left(a_0\cdot 2\psi_{0_\xi}\psi_{1_\xi} + a_1\psi^2_{0_\xi}\right) + 3\delta^2_e a^2_0 a_1\right],\\
N^{III}_{J0} &= \left(a_{0_\tau} + 2a_{0_\xi}\psi_{0_\xi} + a_{0_\xi}\psi_{0_{\xi\xi}}\right),\\
N^{III}_{J1} &= \left[a_{1_\tau}2\left(a_{0_\xi},\psi_{1_\xi} + a_{1_\xi},\psi_{0_\xi}\right) + \left(a_0\psi_{1_{\xi\xi}} + a_1\psi_{0_{\xi\xi}}\right)\right].
\end{aligned}\right\}$$

The attempts to solve the NLS equations like (9) directly would generate for the amplitudes a_n, $n\geq 1$ the secular terms to be proportional to the values $\xi^s,\tau^k,\ s,k=1,2,\ldots$ For this reason we have to arrange the third step of the algorithm on which we would use the new slowly-varying coordinates $\widetilde{X}=\varepsilon\xi, \widetilde{T}=\varepsilon\xi$ and new nonlinear face coordinate

$$\theta\left(\varepsilon,\widetilde{X},\widetilde{T}\right) = \varepsilon^{-1}\theta_0 + \varepsilon^0\theta_1 + \varepsilon^1\theta_2 + \ldots,$$

$\theta_m = \theta_m\left(\widetilde{X},\widetilde{T}\right)$ [3, 9] within the following conditions for their derivatives:

the first order derivatives

$$\left.\begin{aligned}
\theta_\xi &= \varepsilon\theta_{\widetilde{X}} = \nu_0 + \varepsilon\nu_1 + \varepsilon^2\nu_2 + \ldots, \quad \nu_j = \theta_{j\widetilde{X}} = \nu_j\left(\widetilde{X},\widetilde{T}\right),\\
\theta_\tau &= \varepsilon\theta_{\widetilde{T}} = \sigma_0 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \ldots, \quad \sigma_j = \theta_{j\widetilde{T}} = \sigma_j\left(\widetilde{X},\widetilde{T}\right)
\end{aligned}\right\};\tag{13}$$

the second order derivatives

$$\theta_{\xi\tau} = \theta_{\tau\xi} \rightarrow \nu_{j_{\widetilde{T}}} = \sigma_{j_{\widetilde{X}}}, \quad j=0,1,2,\ldots\tag{14}$$

Here in eq. (13) the new unknown functions $\nu_j\left(\widetilde{X},\widetilde{T}\right)$ and $\sigma_j\left(\widetilde{X},\widetilde{T}\right)$ have to be used for eliminating the nonphysical secular parts of the solutions in the higher approximations. In the new variables $\widetilde{X},\widetilde{T},\theta$ the first derivatives $\partial/\partial\tau, \partial/\partial\xi$ became

$$\frac{\partial}{\partial\tau} = \partial^0_\tau + \varepsilon\partial^1_\tau + \varepsilon^2\partial^2_\tau + \ldots, \frac{\partial}{\partial\xi} = \partial^0_\xi + \varepsilon\partial^1_\xi + \varepsilon^2\partial^2_\xi + \ldots,$$

where

$$\left.\begin{aligned}
\partial^0_\tau &= \sigma_0\frac{\partial}{\partial\theta_1}, \quad \partial^1_\tau = \left(\sigma_1\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\widetilde{T}}\right), \quad \partial^2_\tau = \sigma_2\frac{\partial}{\partial\theta};\\
\partial^0_\xi &= \nu_0\frac{\partial}{\partial\theta}, \quad \partial^1_\xi = \left(\nu_1\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\widetilde{X}}\right), \quad \partial^2_\xi = \nu_2\frac{\partial}{\partial\theta}.
\end{aligned}\right\}$$

The second and high order derivatives $\partial^2/\partial\tau\,\partial\xi, \partial^2/\partial\xi^2$ and others have to be defined by using multiplication procedures

$$\frac{\partial^2}{\partial\xi^2} = \frac{\partial}{\partial\xi}\cdot\frac{\partial}{\partial\xi}, \quad \frac{\partial^2}{\partial\xi\partial\tau} = \frac{\partial}{\partial\xi}\cdot\frac{\partial}{\partial\tau} = \frac{\partial}{\partial\tau}\cdot\frac{\partial}{\partial\xi}, \quad \frac{\partial^3}{\partial\xi^3} = \frac{\partial}{\partial\xi}\cdot\frac{\partial^2}{\partial\xi^2},\ldots$$

and for example $\partial^2/\partial\xi^2 = \partial^0_{\xi\xi} + \varepsilon\partial^1_{\xi\xi} + \varepsilon^2\partial^2_{\xi\xi} + \cdots$ where

$$\left.\begin{aligned}
\partial^0_{\xi\xi} &= \nu^2_0\frac{\partial^2}{\partial\theta^2}, \quad \partial^1_{\xi\xi} = \left[2\nu_0\nu_1\frac{\partial^2}{\partial\theta^2} + 2\nu_0\frac{\partial^2}{\partial\theta\partial\widetilde{T}} + \frac{\partial\nu_0}{\partial\widetilde{X}}\frac{\partial}{\partial\theta}\right],\\
\partial^2_{\xi\xi} &= \left(\left(2\nu_0\nu_2 + \nu^2_1\right)\frac{\partial^2}{\partial\theta^2} + 2\nu_1\frac{\partial^2}{\partial\theta\partial\widetilde{X}} + \frac{\partial\nu_1}{\partial\widetilde{X}}\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\widetilde{X}^2}\right)
\end{aligned}\right\}$$

In the new variables any operator $N^\alpha_{R,Jn}$ can be rewritten in the following form

$$N^\alpha_{R,Jn} = N^\alpha_{R,Jn0} + \varepsilon N^\alpha_{R,Jn1} + \varepsilon^2 N^\alpha_{R,Jn2} + \ldots, \quad n=0,1,2.$$

For example, the explicit expressions for the third order operators $N^{III}_{R,J0m}, m = 0,1$ are as follows:

$$\left.\begin{aligned} N^{III}_{R00} &= \left[-a_0\partial^0_\tau\psi_0 + \partial^0_{\xi\xi}\psi_0 + a_0\left(\partial^0_\xi\psi_0\right)^2 + \delta_e^2 a_0^3\right], \\ N^{III}_{J00} &= \left[\partial^0_\tau a_0 + 2\partial^0_\xi a_0\partial^0_\xi\psi_0 + a_0\partial^0_{\xi\xi}\psi_0\right]; \end{aligned}\right\}$$

$$\left.\begin{aligned} N^{III}_{R01} &= \left[-a_0\partial^1_\tau\psi_0 + \partial^1_{\xi\xi}a_0 - 2a_0\partial^0_\xi\psi_0\partial^1_\xi\psi_0\right], \\ N^{III}_{J01} &= \left[\partial^1_\tau a_0 + 2\sum_{j=0}^{1}\partial^j_\xi a_0\partial^{1-j}_\xi\psi_0 + a_0\partial^1_{\xi\xi}\psi_0\right]; \end{aligned}\right\}$$

The explicit expressions for the high operators are much more complicated and for this reason they are not presented in the paper.

Finally we would divide the perturbation phases ψ_j into the two parts $\psi_j = \varepsilon^{-1}\overline{\psi}_j\left(\widetilde{X},\widetilde{T}\right) + \widetilde{\psi}_j\left(\theta,\widetilde{X},\widetilde{T}\right)$, $j = 0,1,2$, where the first part $\overline{\psi}_j$ does not depend on the nonlinear phase θ, but the second one $\widetilde{\psi}_j$ includes the dependence.

3. STATIONARY SOLUTIONS OF THE 3-RD ORDER NLS EQUATIONS

For the basic 3-rd order approximation, the Nonlinear Schrödinger Equations in a real form resulted from the conditions

$$N^{III}_{R00} = N^{III}_{J00} = 0$$

can be written as follows

$$\left.\begin{aligned} &\nu_0^2 a_{0_{\theta\theta}} - a_0\left[\left(\omega_0 + \sigma_0\widetilde{\chi}_0\right) + \left(k_0 + \nu_0\widetilde{\chi}_0\right)^2\right] + \delta_e^2 a_0^3 = 0, \\ &a_{0_\theta}\left[\sigma_0 + 2\nu_0\left(k_0 + \nu_0\widetilde{\chi}_0\right)\right] + \nu_0^2 a_0\widetilde{\chi}_{0_\theta} = 0; \end{aligned}\right\}, \tag{15}$$

where the following variables are denoted

$$\omega_0(X,T) = \overline{\psi}_{0T}, \quad k_0(X,T) = \overline{\psi}_{0X}, \quad \widetilde{\chi}_0 = \widetilde{\psi}_{0\theta};$$

physically ω_0 and k_0 determine the basic linear part in the perturbations of circular frequency and the wave number for the fundamental harmonic in the wave motion, respectively, and $\widetilde{\chi}_0$ is the nonlinear part of the perturbations.

First of all, we would consider the particular case when $\nu_0 = 0$ in the series (13). Then from the condition (14) it follows that $\sigma_0 = 0$ and from the first equation in (15) we would have $\omega_0 = -k_0^2 + \delta_e^2 a_0^2$ and from the second equation results $a_0(\theta) = \mathrm{const}$ respectively. These results describe the well-known periodic Stokes waves of the 3-rd order [14], and because we would consider the general case of modulated wave motion, we suppose that $\nu_0 \neq 0$. In this case eq. (15) can be transformed to the following form

$$\left.\begin{aligned} &a_{\theta\theta} - a\left[\left(\omega_{0_\nu} - \sigma_{0_\nu}\cdot k_{0_\nu}\right) + \left(\sigma_{0_\nu}\widetilde{\chi} + \widetilde{\chi}^2\right)\right] + \delta_\nu^2 a_s^2\cdot a^3 = 0, \\ &a_\theta\left(\sigma_{0_\nu} + 2\widetilde{\chi}\right) + a\cdot\widetilde{\chi}_\theta = 0, \end{aligned}\right\}, \tag{16}$$

where $a, \widetilde{\chi}$ are the modified unknown variables according to the relations

$a_0 = a_s\cdot a,\ \widetilde{\chi} = \left(\widetilde{\chi}_0 + k_0/\nu_0\right),\ a_s = a_s(X,T)$ is some characteristic value of the amplitudes and the following relations are denoted

$$\omega_{0_\nu} = \omega_0/\nu_0^2,\ \sigma_{0_\nu} = \sigma_0/\nu_0^2,\ k_{0_\nu} = k_0/\nu_0,\ \delta_\nu^2 = \delta_e^2/\nu_0^2.$$

For the solution of the eq. (16) we would start from the simplified case when $\widetilde{\chi}_{0_\theta} = 0$ and correspondingly $\widetilde{\psi}_0(\theta)\sim\theta$. In particular it means that phase $\widetilde{\psi}_0$ can be formally included into the phase $\overline{\psi}$ in this case. Then from the second equation in (16) it follows that $\widetilde{\chi}_0 = -\left(k_{0_\nu} + \sigma_{0_\nu}/2\right)$ and the first equation can be transformed to the resulting equation for the amplitudes envelope in the canonical form

$$\left(\frac{da}{d\vartheta}\right)^2 = \left(a^2 - a_1^2\right)\left(a_2^2 - a^2\right), \quad a\in\left[a_1; a_2\right], \tag{17}$$

where $\vartheta = \delta_\nu a_s\cdot\theta/\sqrt{2}$ is a modified phase coordinate and coefficients in the equation are as follows $a_1^2 = \widetilde{\Omega}_0\cdot R/2, \quad a_2^2 = \widetilde{\Omega}_0(1 - R/2)$. Here the next values are denoted too

$$\widetilde{\Omega}_0 = 2\Omega_0/\delta_\nu^2 a_s^2$$

$$\Omega_0 = \left[\omega_{0_\nu} - \sigma_{0_\nu}\left(k_{0_\nu} + \sigma_{0_\nu}/4\right)\right],$$

$$R = \left[1 - \left(1 - 2c_0\delta_\nu^2 a_s^2/\Omega_0^2\right)^{1/2}\right]$$

and $c_0 \geq 0$ is a positive integrating constant.

Eq. (17) has the solution in the terms of Jacobe functions — so-called delta-function $dn(\vartheta;m)$, where $m \in [0;1]$ is the square of modulus of the function. Thus we can write the solution in the resulting form

$$\left.\begin{array}{l} a(\vartheta) = dn(\vartheta;m), \quad a \in \left[(1-\delta a/a_s);1\right]; \\ \Omega_0 = \delta_\nu^2 a_s^2 (1-m/2), \quad c_0 = \delta_\nu^2 a_s^2(1-m)/2; \\ m = \left[1-(1-\delta a/a_s)^2\right] \end{array}\right\}, \quad (18)$$

where values a_S and δa are the maximum amplitude and the depth of the amplitude modulation in the wave motion, respectively.

There are two well-known asymptotical cases for the solution (18) [15]: 1) stationary Stokes waves with $a(\vartheta) = \text{const}$ for the case when m = 0 and 2) soliton solution with $a(\vartheta) \sim ch^{-1}(\vartheta)$ for the case when m = 1. The solution $a(\vartheta) = dn(\vartheta;m)$ is a periodically modulated solution for the envelope of wave amplitudes in wave groups.

The generalization of the solution (18) follows from the assumption that nonlinear part of the perturbation phase $\tilde{\psi}_0(\theta) \neq 0$ and than $\chi_0(\theta) \neq \text{const}$. In that the most general case the second equation in the system (16) can be integrated with the results

$$\left.\begin{array}{l} \tilde{\chi}(\theta) = -\dfrac{1}{2}\sigma_{0_\nu} + c_1 a^{-2}, \\ \tilde{\psi}_0(\theta) = \tilde{\psi}_0^0 - \dfrac{1}{2}\sigma_{0_\nu}\theta + c_1\displaystyle\int_0^\theta a^{-2}(\theta)d\theta, \end{array}\right\} \quad (19)$$

where $\tilde{\psi}_0^0$ and c_1 are arbitrary integration constants.

Multiplying the first equation in the system (16) by the value a_θ an then integrating it one time allows us to obtain the following equation for the new modified variable $q_0 = \alpha - a^2$ in the canonical form

$$\left(\frac{dq_0}{d\vartheta}\right)^2 = 4q_0^3 - g_2 q_0 - g_3, \quad (20)$$

where the coefficients (so called invariants) g_2, g_3 and value α are described by the formulae

$$g_2 = \left(\frac{1}{27}\frac{\Omega_0^2}{\delta_\nu^4 a_s^4} - \frac{8c_0}{\delta_\nu^2 a_s^2}\right), \quad g_3 = 8\left(-\frac{8}{27}\frac{\Omega_0^3}{\delta_\nu^6 a_s^6} + \frac{2}{3}\frac{\Omega_0 c_0}{\delta_\nu^4 a_s^4} - \frac{c_1^2}{\delta_\nu^2 a_s^2}\right), \ \alpha = 2\Omega_0/\left(3\delta_\nu^2 a_s^2\right).$$

Note that for the particular case when $c_1 \equiv 0$ and $\tilde{\psi}_0(\theta) \sim 0$ eq. (20) transforms to the equation (17) within the following expressions for the values α, g_2 and g_3

$$\left.\begin{array}{l} \alpha = \dfrac{2}{3}\left(1 - \dfrac{1}{2}m\right), \quad g_2 = \dfrac{4}{3}\left(1 - m + m^2\right) \in [1;4/3], \\ g_3 = -\dfrac{8}{27}\left(1 - \dfrac{1}{2}m\right)(2m-1)(m+1) \in [-8/27;+8/27] \end{array}\right\}$$

Wejershtruss function $\wp(\nu; g_2, g_3)$ [1, 2] is the solution of the eq. (20) thus

$$q_0(\vartheta) = \wp(\vartheta; g_2, g_3) = \wp(\vartheta; e_1, e_2, e_3), \quad (21)$$

where e_j, j = 1,2,3 are the roots of the polynomial in the right hand of eq. (20). These roots have to be defined according to the rule

$$|e_1| \leq |e_2|, \quad e_3 = -(e_1 + e_2)$$

by choosing from the following three values

$$\left.\begin{array}{l} e_0 = \sqrt{\dfrac{g_2}{3}} \cdot \cos\gamma, \quad e_+ = \sqrt{\dfrac{g_2}{3}} \cdot \cos\left(\gamma + \dfrac{2\pi}{3}\right), \\ e_- = \sqrt{\dfrac{g_2}{3}} \cdot \cos\left(\gamma - \dfrac{2\pi}{3}\right), \quad \gamma = \dfrac{1}{3}\arccos\left(\dfrac{3g_3}{\sqrt{g_2^3}}\right) \end{array}\right\}, \quad (22)$$

The solution (21), (22) would be real when the following condition would take place

$$\Delta = g_2^3 - 27g_3^2 = 16(e_2 - e_3)^2(e_3 - e_1)^2(e_1 - e_2)^2 \geq 0.$$

The finite solutions satisfy the two combinations of the roots

1) $e_2 \ge e_1 \ge 0, e_3 \le 0, \quad \wp \in [e_3; e_1]$;

2) $e_2 < e_1 < 0, e_3 > 0, \quad \wp \in [e_2; e_1]$,

where in the first case the roots would be

$$e_3 = -e_a \cos\gamma_3 \le 0: \quad |\gamma_3| \le \frac{\pi}{6};$$
$$e_1 = e_a \cdot \sin\left(\frac{\pi}{6} - |\gamma_3|\right), \quad e_2 = |e_3| - e_1 \quad (23)$$

and in the second case respectively

$$e_3 = e_a \cos\gamma_3 \ge 0; \quad |\gamma_3| \le \frac{\pi}{6};$$
$$e_1 = -e_a \cdot \sin\left(\frac{\pi}{6} - |\gamma_3|\right), \quad e_2 = -e_3 + |e_1|. \quad (24)$$

In eq. (23) and (24), $e_a = \widetilde{\Omega}_0 \sqrt{\lambda}$ and angle γ_3 have to be determined from the equation $4\cos^3\gamma_3 - 3\cos\gamma_3 = \chi/\sqrt{\lambda^3}$, where

$$\chi = \left[\left(\xi_1^2 - 1\right) + 2\xi_0/3\right] \in [-1;+1],$$
$$\lambda = (1 - \xi_0) \in [0;+1]; \; \xi_1 = \widetilde{c}_1^2/\widetilde{\Omega}_0^3,$$
$$\xi_0 = \widetilde{c}_0/\widetilde{\Omega}_0^3$$

and

$$\widetilde{c}_1^2 = \frac{8c_1^2}{\delta_\nu^2 a_s^2}, \quad \widetilde{c}_0 = \frac{8c_0}{\delta_\nu^2 a_s^2}, \quad \widetilde{\Omega}_0 = \frac{4\left(\omega_{0_\nu}^2 + k_{0_\nu}^2\right)}{3\delta_\nu^2 a_s^2}$$

The domain of the real solutions is showed on the diagram in Figure 1.

The generalized stationary solution of the basic NLS equation of the 3-rd order has been derived in the terms of Wejershtruss elliptical function $\wp(\vartheta; g_2, g_3) = \wp\left(\vartheta; \{e_j\}_1^3\right)$. So for the amplitude envelope we would have

$$a = a_s\sqrt{a_0 - \wp(\vartheta)}, \quad (25)$$

where a_S is a characteristic value of the amplitudes, $a_0 = 2\Omega_0/\left(3\delta_\nu^2 a_s^2\right)$.

From the practical point of view it is better to operate in the calculations Jacobi elliptical functions (*sn*, *cn*, *dn*) which are finite and satisfy the following differential equations [1, 2]

$$\left(\frac{\partial sn}{\partial \nu}\right)^2 = \left(1 - sn^2\nu\right)\left(1 - m \cdot sn^2\nu\right),$$
$$\left(\frac{\partial cn}{\partial \nu}\right)^2 = \left(1 - cn^2\nu\right)\left[(1 - m) + m \cdot cn^2\nu\right], \quad (26)$$
$$\left(\frac{\partial dn}{\partial \nu}\right)^2 = \left(1 - dn^2\nu\right)\left[dn^2\nu - (1 - m)\right],$$

where $m \in [0;1]$ is a square of modules of Jacobi elliptical functions and it is denoted $\nu = \vartheta\sqrt{e_1 - e_3}$. The additional relations between elliptical functions are as follows [2]

$$cn^2\nu + sn^2\nu = msn^2\nu + dn^2\nu =$$
$$= m\left(1 - cn^2\nu\right) + dn^2\nu = 1.$$

To agree the canonical differential equations (20) and (26), we would write the following transformations

$$\wp(\vartheta) = \gamma_0 + \gamma_2 J^2(\nu), \quad \vartheta = \gamma_1 \nu$$

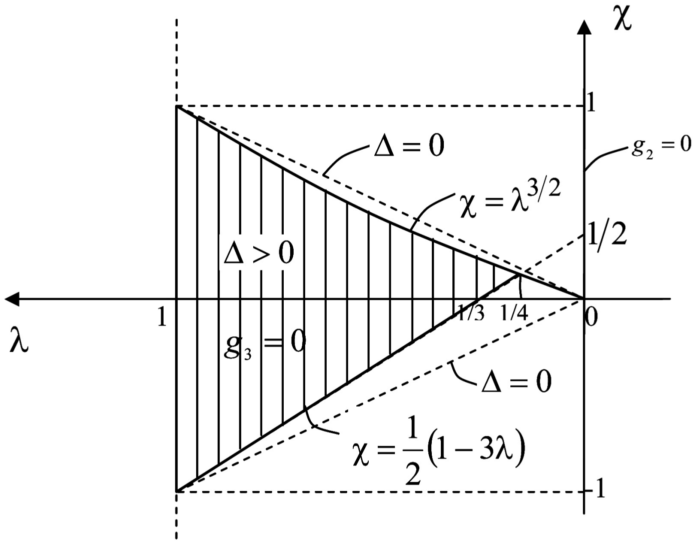

Figure 1.

where J = (sn, cn, dn) and coefficients γ_n, n = 0, 1, 2 have to be determined from the agreement. Resulting formulae have been derived in the form

$$\begin{aligned}\wp(\vartheta) &= e_3 + (e_2 - e_3)sn^2\left(\sqrt{e_1 - e_3}\,\vartheta\right) = \\ &= e_2 + (e_2 - e_3)cn^2\left(\sqrt{e_1 - e_3}\,\vartheta\right) = \qquad (27) \\ &= e_1 + (e_1 - e_3)dn^2\left(\sqrt{e_1 - e_3}\,\vartheta\right).\end{aligned}$$

In this context, the values of the polynomial roots $\{e_n\}$, discriminate Δ and invariants q_2, q_3 are related with the square of modules m and the new parameter $k = 2K(m)/\omega_1$ (where $K(m)$ is an elliptical integral) by the formulae

$$\begin{gathered} e_1 = \frac{1}{3}k^2(2 - m), \quad e_2 = \frac{1}{3}k^2(2m - 1), \\ e_3 = -\frac{1}{3}k^2(1 + m); \\ \Delta = 16k^{12}m^2(m - 1`)^2, \quad m \in [0;1]; \qquad (28) \\ q_2 = \frac{4}{3}k^4(m^3 + 1`)(m + 1`)^{-1}, \\ q_3 = \frac{4}{27}k^5(m + 1`)(m - 2`)(2m - 1). \end{gathered}$$

From the formulae in eq. (28) it follows that condition Δ = 0 (boundary conditions for the real solutions) is possible when $m = 0$ (the value of m for periodic Stokes waves) and when $m = 1$ (the value of m for the envelope soliton solution). Condition q_3 = 0 is only possible when $m = ½$ for $m \in [0;1]$ and finally condition $q_2 > 0$ is satisfied for all values of $m \in [0;1]$.

CONCLUSIONS

1. Weakly and moderately nonlinear wave motions with narrow bend spectrum – modulated wave groups motions – can be effectively studied by using the so called Nonlinear Schrödinger equations of different orders. These equations, as governing equations (NLS) for the complex amplitude of fundamental mode in the wave field, are resulted from the asymptotic solution of the boundary-value problem for nonlinear surface water waves. 2. The attempts to solve these equations in the forth and high order displayed the generation of the non-physical secular terms in the solution for amplitude envelope of the waves. For these reasons special modifications and generalizations of the traditional perturbation technique have to be applied to eliminate such terms from the solution. The generalized multi-scale technique including nonlinear phase coordinate in the perturbations can be considered as the appropriate instrument for the suppressing of the secular terms. 3. For the basic 3-rd order NLS the most general stationary solution have been derived in the terms of elliptical functions.

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