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# NONLINEAR ASYMPTOTICAL ESTIMATIONS OF THE SHORT-TERM PROBABILITY DISTRIBUTIONS FOR STEEP WIND-GENERATED WAVES

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**Abstract.** The paper deals with the generalizations of short-term distributions of wind-generated wave elevations in sea conditions by using characteristic functions technique and corresponding asymptotic Gram–Charlier–Edgeworht's sets. The generalization includes nonlinear effects into the Gaussian distributions. According to the approach, cumulants up to 12-th order have been derived and its numerical estimations have been done by using nonlinear wave-group model of the 6-th order for energetic component of irregular wind-generated wave field.

**Keywords:** wind-generated waves, statistical distributions, nonlinear effects, Gram-Charlier-Edgeworht's sets.

**Анотація.** Розглянуто нелінійне узагальнення короткотермінових розподілів для ординат вітрових хвиль у штормі із застосуванням техніки характеристичних функцій та асимптотичних рядів типу Грама–Шарльє–Еджворта для екстремальних значень хвильових ординат. Кумулянти в асимптотичному ряді отримано до дванадцятого порядку, а їх числову оцінку виконано із залученням нелінійної групової моделі хвиль Стокса шостого порядку.

**Ключові слова:** вітрові хвилі, статистичні розподіли, нелінійні ефекти, ряди Грама-Шарльє-Еджворта.

**Аннотация.** Рассмотрено нелинейное обобщение кратковременных распределений для ординат ветровых волн в шторме на основе использования техники характеристических функций и соответствующих асимптотических рядов типа Грама–Шарлье–Эджворта для экстремальных значений волновых ординат. Кумулянты в асимптотическом ряде получены до двенадцатого порядка, а их численная оценка выполнена с использованием нелинейной групповой модели волн Стокса шестого порядка.

**Ключевые слова:** ветровые волны: статистические распределения, нелинейные эффекты: ряды Грама-Шарлье-Эджворта.

## INTRODUCTION

The traditional point of view on the short-term statistics of irregular wind-generated waves is based on Gaussian well-known probability distribution for vertical wave elevations $\zeta_w$ and on Rayleigh probability distribution for wave heights $h_w$ [1, 2, 6, 9]. These distributions have been applied very widely in the solution of different sea-keeping problems, in the estimations of ship strength in real sea conditions, in ship design procedures etc. Gaussian probability distribution function for wave elevations $\zeta_w$ is as follows:

$$p^{Ga}(\zeta_w) = \frac{1}{\sqrt{2\pi}\sigma_\zeta}\exp\left(-\frac{\zeta_w^2}{2\sigma_\zeta{}^2}\right), \tag{1}$$

and Rayleigh probability distribution function for wave heights hw has the following form

$$p^{Ra}(h_w) = \frac{h_w/2}{\sigma_\zeta{}^2} \exp\left( -\frac{(h_w/2)^2}{2\sigma_\zeta{}^2} \right),\ h_w > 0, \qquad (2)$$

where $\sigma_\zeta = \sqrt{D_\zeta}$ is the r.m.s. value of wave elevations.

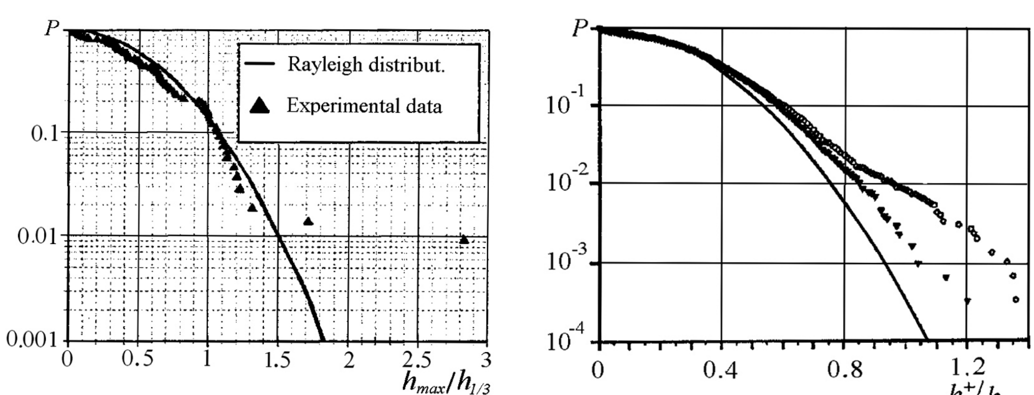
Gaussian and Rayleigh probability distributions are based on the assumption that irregular wave motions are linear physical processes and for Rayleigh probability distribution there is an additional assumption that the wave frequency spectrum $S_w(\sigma)$ is narrowband [1, 2, 6].

For the lust two decades the satellite monitoring of the World Ocean surface has been used very extensively and the results obtained showed that for extreme waves Gaussian and Rayleigh probability distributions are not correct [4, 5, 10]. See, for example, the distributions in Fig. 1, *a* for wave crest elevations $h^+$ and Fig. 1, *b* for extreme wave heights $h_{\max}$. This Fig. displays that, for extreme waves, real statistical distributions are significantly greater than it follows from the distributions (1) and (2). According to the latest statistical data, extreme waves have significantly high levels of the probabilities than it follows from the classical distributions (1) and (2).

This fact seems very important for the safety of ships in extreme sea conditions, because extreme waves may cause serious damages of ship structures (see Fig. 2, *a* for aircraft carrier and Fig. 2, *b* for tanker).

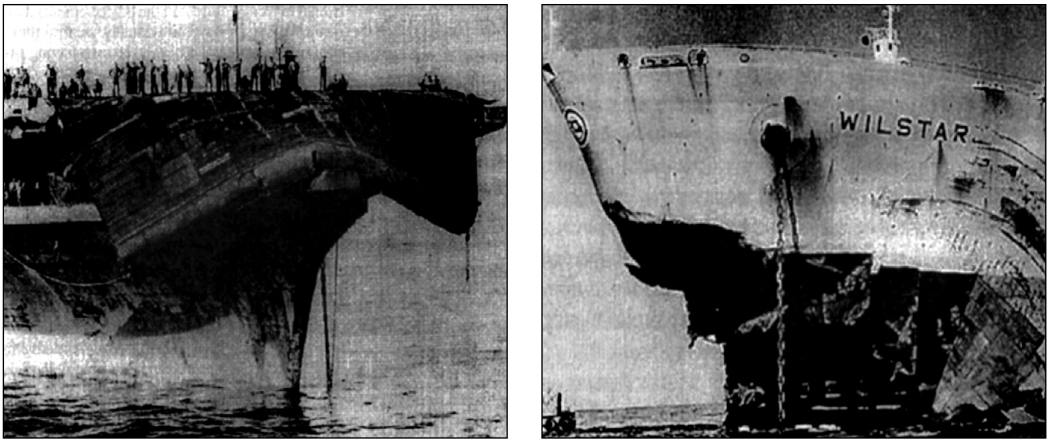
Therefore further research work in the investigations of real statistics of windgenerated waves, especially in extreme sea conditions, is very vitally needed. The investigations can be arranged on the basic of information obtained from special satellite equipments and systematic monitoring of the World Ocean surface and from the other hand on the advanced theoretical investigations in this traditional area of probabilistic theory. The main idea is to include the nonlinear effects in the wave surface, which take place for the extreme waves, into the theoretical probabilistic models for main wave characteristics — wave elevations, wave amplitudes, wave heights and periods.

This paper deals with the nonlinear generalizations of short-term distributions of windgenerated wave elevations in real sea conditions by using characteristic function technique and corresponding asymptotic Gram-Edgeworht's sets. The traditional approach is considered in beginning of the paper, but the alternative approach based on the modification of the traditional technique is discussed late. The second part of the paper deals with the theoretical estimations of the cumulants of the characteristic function and comparisons of the results obtained with the independent experimental data




*a* *b*

**Fig. 1.** Comparisons of Rayleigh probability distribution (solid line) with the experimental data (symbols): *a* — laboratory data; *b* — field data



*a* *b*

**Fig. 2.** Damages of ship structures for: *a* — aircraft carrier; *b* — tanker

are considered too. Finally the problem of the convergence of the asymptotical sets is briefly discussed.

## 1. CHARACTERISTIC FUNCTION TECHNIQUE

For the purpose of nonlinear estimation of short-term marginal probability distribution of wave elevations here we would use the socalled characteristic functions technique [3, 6]. According to the technique probability distribution function $p(\zeta_w)$ and characteristic function $\Phi(k)$ are related by the following Fourier-transformation

$$\left.\begin{aligned} p(\zeta_w) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\Phi(k)\exp(-ik\zeta_w)dk, \\ \Phi(k) &= \int\limits_{-\infty}^{\infty}p(\zeta_w)\exp ik\zeta_w d\zeta_w \end{aligned}\right\}, \qquad (3)$$

where $k$ is the real parameter of the characteristic function.

To estimate the characteristic function in the eq. (3), we would expand the exponential function $\exp ik\zeta_w$ in the power series of $k^n$

$$\exp ik\zeta_w = 1 + \sum_{n=1}^{N}\frac{1}{n!}(ik\zeta_w)^n, \; N >> 1,$$

so after integrating in eq. (3) the characteristic function $\Phi(k)$ becomes:

$$\Phi(k) = 1 + \sum_{n=1}^{N}\frac{i^n}{n!}k^n \cdot \mu_n, \; \mu_n = \int\limits_{-\infty}^{\infty}\zeta_w^n p(\zeta_w)d\zeta_w, \qquad (4)$$

where $\mu_n$ are the $n$-th order moments of the distribution function $p(\zeta_w)$. These moments can be expressed in the terms of the moments taken respectively to the mean value (or central moments) $\overline{\mu}_n$ of the distribution function by the following relations:

$$\mu_n = \sum_{j=0}^{n}C_n^j m_\zeta{}^j \overline{\mu}_{n-j}, \quad \overline{\mu}_n = \int\limits_{-\infty}^{\infty}(\zeta_w - m_\zeta)^n p(\zeta_w)d\zeta_w, \quad n = 2,3,\ldots,N,$$

where $m_\zeta \equiv \mu_1$ is the mean value of wave elevations and this value is typically very small for wind-generated waves in real sea conditions [1, 2, 6].

According to the traditional procedures [1, 3], the power series for the characteristic function in eq. (4) are transformed into the following exponential form

$$\Phi(k) = \exp\left(\lambda_1\frac{i}{1!}k + \lambda_2\frac{i^2}{2!}k^2 + \lambda_3\frac{i^3}{3!}k^3 + \mathrm{K} + \lambda_m\frac{i^m}{m!}k^m\right), \qquad (5)$$

where $\lambda_j$ is the $j$-th order cumulants of the characteristic function. For the estimations of cumulants $\lambda_j$ in terms of moments $\mu_n$, we need to expand the expression (5) into the power series and than to compare this series with the series (4). Finally, after a number of laborious transformations, these expressions for the first eight cumulants are as follows:

$$\begin{aligned}&\lambda_1 = m_\zeta \approx 0;\quad \lambda_2 = \bar{\mu}_2;\quad \lambda_3 = \bar{\mu}_3;\quad \lambda_4 = \bar{\mu}_4 - 3\bar{\mu}_2^2;\quad \lambda_5 = \bar{\mu}_5 - 10\bar{\mu}_2\bar{\mu}_3;\\ &\lambda_6 = \bar{\mu}_6 - \left(15\bar{\mu}_2\bar{\mu}_4 + 10\bar{\mu}_3^2\right) + 30\bar{\mu}_2^3;\quad \lambda_7 = \bar{\mu}_7 - \left(21\bar{\mu}_2\bar{\mu}_5 + 35\bar{\mu}_3\bar{\mu}_4\right) + 210\bar{\mu}_2^2\bar{\mu}_3;\\ &\lambda_8 = \bar{\mu}_8 - \left(28\bar{\mu}_2\bar{\mu}_6 + 56\bar{\mu}_3\bar{\mu}_5 + 35\bar{\mu}_4^2\right) + \left(420\bar{\mu}_2^2\bar{\mu}_4 + 560\bar{\mu}_3^2\bar{\mu}_2\right) - 630\bar{\mu}_2^4.\end{aligned} \tag{6}$$

These results correspond to the well-known expressions for the cumulants, see, for example, publication [3]. Another important conclusion which is made from the eq. (6) is that the formulae for the cumulants do not include the first moment — mean elevation $m_\zeta \equiv \mu_1$. Hence we may ignore the mean elevation of the irregular wind-generated waves in the future and only consider central moments of the distribution function. A number of minor modifications of the traditional procedures considered above allow us to obtain additional four cumulants up to the 12-th order. Fist of all, we rewrite eq. (4) in the following way:

$$\Phi(k) = \exp\left[\ln\left(1 + \sum_{n=1}^{N} \frac{i^n}{n!} k^n \cdot \mu_n\right)\right]$$

and then expand logarithmic function in this expression into the power series of $k^n$. Finally the cumulants $\lambda_j$, $j = 9...12$, have been derived as follows:

$$\left.\begin{aligned}\lambda_9 = &\left[\bar{\mu}_9 - \left(36\bar{\mu}_2\bar{\mu}_7 + 84\bar{\mu}_3\bar{\mu}_6 + 126\bar{\mu}_4\bar{\mu}_5\right) + \left(756\bar{\mu}_2^2\bar{\mu}_5 + 2520\bar{\mu}_2\bar{\mu}_3\bar{\mu}_4 + \right.\right.\\ &\left.\left. + 560\bar{\mu}_3^3\right) - 7560\bar{\mu}_2^3\bar{\mu}_3\right];\\ \lambda_{10} = &\left[\bar{\mu}_{10} - \left(45\bar{\mu}_2\bar{\mu}_8 + 120\bar{\mu}_3\bar{\mu}_7 + 210\bar{\mu}_4\bar{\mu}_6 + 126\bar{\mu}_5^2\right) + \left(1260\bar{\mu}_2^2\bar{\mu}_6 + 4200\bar{\mu}_3^2\bar{\mu}_4 + \right.\right.\\ &\left.\left. + 3150\bar{\mu}_4^2\bar{\mu}_2 + 5040\bar{\mu}_2\bar{\mu}_3\bar{\mu}_5\right) - \left(18900\bar{\mu}_2^3\bar{\mu}_4 + 37800\bar{\mu}_2^2\bar{\mu}_3^2\right) + 22680\bar{\mu}_2^5\right];\\ \lambda_{11} = &\left[\bar{\mu}_{11} - \left(55\bar{\mu}_2\bar{\mu}_9 + 165\bar{\mu}_3\bar{\mu}_8 + 330\bar{\mu}_4\bar{\mu}_7 + 462\bar{\mu}_5\bar{\mu}_6\right) + \left(1980\bar{\mu}_2^2\bar{\mu}_7 + \right.\right.\\ &\left. + 9240\bar{\mu}_3^2\bar{\mu}_5 + 13860\bar{\mu}_2\bar{\mu}_4\bar{\mu}_5\right) - \left(41580\bar{\mu}_2^3\bar{\mu}_5 + 92400\bar{\mu}_2\bar{\mu}_3^3 + \right.\\ &\left.\left. + 207900\bar{\mu}_2^2\bar{\mu}_3\bar{\mu}_4\right) + 415880\bar{\mu}_2^4\bar{\mu}_3\right];\\ \lambda_{12} = &\left[\bar{\mu}_{12} - \left(66\bar{\mu}_2\bar{\mu}_{10} + 220\bar{\mu}_3\bar{\mu}_9 + 495\bar{\mu}_4\bar{\mu}_8 + 792\bar{\mu}_5\bar{\mu}_7 + 462\bar{\mu}_6^2\right) + \right.\\ &+ \left(2970\bar{\mu}_2^2\bar{\mu}_8 + 18480\bar{\mu}_3^2\bar{\mu}_6 + 11550\bar{\mu}_4^3 + 16632\bar{\mu}_5^2\bar{\mu}_2 + 15840\bar{\mu}_2\bar{\mu}_3\bar{\mu}_7 + \right.\\ &\left. + 27720\bar{\mu}_2\bar{\mu}_4\bar{\mu}_6 + 55440\bar{\mu}_3\bar{\mu}_4\bar{\mu}_5\right) - \left(83160\bar{\mu}_2^3\bar{\mu}_6 + 311850\bar{\mu}_2^2\bar{\mu}_4^2 + \right.\\ &\left. + 498960\bar{\mu}_2^2\bar{\mu}_3\bar{\mu}_5 + 831600\bar{\mu}_3^2\bar{\mu}_2\bar{\mu}_4\right) + \left(1247400\bar{\mu}_2^4\bar{\mu}_4 + 3326400\bar{\mu}_2^3\bar{\mu}_3^2\right) -\\ &\left. - \left(1247400\bar{\mu}_2^6 + 92400\bar{\mu}_3^4\right)\right].\end{aligned}\right\} \tag{7}$$

As it follows from the eq. (7), the complexity of the expressions for the high cumulants and the numerical values of the coefficients in the expressions grow considerably. And it is important to note that for Gaussian distribution even moments become as $\bar{\mu}_{2n} = (2n)!(\bar{\mu}_2)^n/n!$, all odd moments torn to zero $\bar{\mu}_{2n+1} \equiv 0$ and correspondingly all cumulants $\lambda_j j > 2$, are totally eliminated.

After deriving the cumulants of the characteristic function $\Phi(k)$ in some approach, the eq. (4) has to be substituted into the Fourier-transform (1) and it leads finally to the following expression for the distribution function $p(\zeta_w)$

$$p(\zeta_w) = p^{Ga}(\zeta_w) \cdot I(\zeta_w, \lambda_j), \tag{8}$$

where correction factor $I(\zeta_w, \lambda_j)$ includes nonlinear effects in the wave elevations. This nonlinear correction factor has been derived in the form

$$I = \left[1 + \sum_{n=3}^{12} \frac{1}{n!} \mathrm{P}_n\left(\tilde{\lambda}_j\right) \cdot E_n\left(\tilde{\zeta}\right)\right], \tag{9}$$

where $\mathrm{P}_n\left(\tilde{\lambda}_j\right)$ and $E_n\left(\tilde{\zeta}\right)$ are the algebraic polynomials of values $\tilde{\lambda}_j = \lambda_j / \sigma_\zeta{}^j$ and $\tilde{\zeta} = \zeta_w / \sigma_\zeta$, respectively. The explicit formulae for the polynomials are

$$E_n(\tilde{\zeta}) = \tilde{\zeta}^n + \sum_{m=1,2,K}^{n/2} \frac{(-1)^m (n)!}{2^m m!(n-2m)!} \tilde{\zeta}^{n-2m}, \quad n \geq 3 \tag{10}$$

and

$$\left.\begin{aligned}
&P_3 = \tilde{\lambda}_3; \quad P_4 = \tilde{\lambda}_4; \quad P_5 = \tilde{\lambda}_5; \quad P_6 = \left(\tilde{\lambda}_6 + 10\tilde{\lambda}_3^2\right), \quad P_7 = \left(\tilde{\lambda}_7 + 35\tilde{\lambda}_3\tilde{\lambda}_4\right); \\
&P_8 = \left(\tilde{\lambda}_8 + 56\tilde{\lambda}_3\tilde{\lambda}_5 + 35\tilde{\lambda}_4^2\right), \quad P_9 = \left[\tilde{\lambda}_9 + \left(84\tilde{\lambda}_3\tilde{\lambda}_6 + 126\tilde{\lambda}_4\tilde{\lambda}_5\right) + 280\tilde{\lambda}_3^3\right]; \\
&P_{10} = \left[\tilde{\lambda}_{10} + \left(120\tilde{\lambda}_3\tilde{\lambda}_7 + 210\tilde{\lambda}_4\tilde{\lambda}_6 + 126\tilde{\lambda}_5^2\right) + 2100\tilde{\lambda}_4\tilde{\lambda}_3^2\right]; \\
&P_{11} = \left[\tilde{\lambda}_{11} + \left(165\tilde{\lambda}_3\tilde{\lambda}_8 + 330\tilde{\lambda}_4\tilde{\lambda}_7 + 462\tilde{\lambda}_5\tilde{\lambda}_6\right) + \left(4620\tilde{\lambda}_3^2\tilde{\lambda}_5 + 5775\tilde{\lambda}_4^2\tilde{\lambda}_3\right)\right]; \\
&P_{12} = \left[\tilde{\lambda}_{12} + \left(220\tilde{\lambda}_3\tilde{\lambda}_9 + 495\tilde{\lambda}_4\tilde{\lambda}_8 + 792\tilde{\lambda}_5\tilde{\lambda}_7 + 462\tilde{\lambda}_6^2 + \left(9240\tilde{\lambda}_3^2\tilde{\lambda}_6 + \right.\right.\right. \\
&\qquad\left.\left.\left. + 27720\tilde{\lambda}_3\tilde{\lambda}_4\tilde{\lambda}_5 + 5775\tilde{\lambda}_4^3\right) + 15400\tilde{\lambda}_3^4\right)\right].
\end{aligned}\right\} \tag{11}$$

## 2. EVALUATION OF THE CUMULANTS

Resulting eq. (9)–(11) show that nonlinear correction factor depends on the values of the cumulants of the characteristic function. There are two ways for evaluating of the cumulants and moments, respectively: namely, the first way is based on the experimental estimations for a number of irregular wave trains to be generated in a wave tank or recorded in real sea conditions, and the second one is related with the theoretical estimations of the moments by using appropriate nonlinear hydrodynamic models for irregular wave elevations. The first approach has been realized in NASA Wallops Flight Center in the USA and the results for the first eight cumulants were presented in the publication [3]. The main ideas in the second approach have been discussed by M.S. Longuet-Higgins and M.A. Srokosz in their paper [8] for the simple wave model and the first three cumulants being considered. The results of the experimental investigations are very important from the point of view of verifying the theoretical results, but having the further research in view, we would like to consider the theoretical approach starting from generalized nonlinear model for wave elevations.

Namely, we would consider here the nonlinear wave-group model of the sixth order on wave steepness for the main energy-carrying component of irregular wave motion

$$\tilde{\zeta}_w(x,t) = \sum_{n=1}^{6} a_n(\varepsilon x, \varepsilon t)\cos n\theta(x,t,\varepsilon), \tag{12}$$

where $a_n$, $n \geq 2$ are the slowly-varying amplitudes of high bounded harmonics in the wave profile which depend on the amplitude of the first fundamental harmonic in the wave motion $a_1 \equiv a$; $\theta(x,t,\varepsilon)$ is the slowly-varying phase coordinate to be determined as

$$\theta(x,t,\varepsilon) = k_m x + \sigma_m t + \varepsilon\Delta\theta(\varepsilon x, \varepsilon t),$$

$k_m = 2\pi/\lambda_m$, $\sigma_m = \sqrt{gk_m}$ and $\lambda_m$ is considered as the average wave length of the energy-carrying component of irregular waves, e.g. in the vicinity of spectral function's maximum; and, finally, $0 << \varepsilon < 1$ is a small parameter describing slowly-varying modulations in the wave motion.

Amplitudes of the harmonics in eq. (12) have been derived by using the perturbation technique for solving a nonlinear boundary-value problem for surface waves [7] and these amplitudes can be written here in the following form

$$a_n = a_w \delta_w^{\,n-1} \nu_n^o \left(1 + \nu_n' \delta_w^2 + \nu_n'' \delta_w^4 + O\left(\delta_w^6\right)\right), \qquad n = 1...6,$$

where $a_w = h_w/2$ is the slowly-varying amplitude envelope of irregular waves; $\delta_w = k_m a_w = \pi h_w/\lambda_w$ is the slowly-varying steepness of wave slopes and numerical factors $\nu_n^o, \nu_n', \nu''$ in the 6-th order nonlinear wave model are as follows [7]:

$$\{\nu_n^o\}_1^6 = \{1; 1/2; 3/8; 1/3; 125/386; 27/80\}, \ \{\nu_n'\}_1^6 = \{-3/8; 2/3; 2; 127/60; 0; 0\},$$

$$\{\nu_n''\}_1^6 = \{211/192; -11/24; 0; 0; 0; 0\}.$$

Note that inclusion of more high harmonics with $n > 6$ into the series (12) for the wave surface adds insignificant modifications in the wave profile and only in the near vicinity of the wave crests. The steepness of wave slopes $\delta_w$ related with the steepness of the slopes of the first fundamental harmonics $\delta_a = k_m a$ in wave motion by the following relation

$$\delta_w = \delta_a + \tfrac{3}{8}\delta_a^3 - \tfrac{2}{3}\delta_a^5 + O\left(\delta_a^7\right).$$

The same form has the relation between slowly-varying amplitudes $a_w$ and $a$, respectively.

From the other hand, we would suppose that variables $a$ and $\theta$ are statistically independent and these variables have the following distribution function

$$p(a,\theta) \approx p(a)p(\theta) = \frac{1}{2\pi}\frac{a}{\sigma_\zeta^2}\exp\left(-\frac{a}{2\sigma_\zeta^2}\right), \quad \theta \in [0;2\pi], \quad a \in [0;\infty].$$

Then the moments of the distribution $\overline{\mu}_n$ may be evaluated directly from the basic formulation

$$\overline{\mu}_n = \int_{-\infty}^{\infty}\left(\zeta_w - m_\zeta\right)^n p(\zeta_w)d\zeta_w = \int_0^{\infty}\int_0^{2\pi}\zeta_w(a,\theta)^n p(a,\theta)da d\theta. \tag{13}$$

After applying the integration procedures in the eq. (13), the resulting expressions for the moments $\overline{\mu}_n$ have been derived in the following form:

$$\left.\begin{aligned} &\overline{\mu}_j = m_j^o\delta_s^\alpha\left(1 + m_j'\delta_s^2 + m_j''\delta_s^4 + O\left(\delta_s^6\right)\right), \; j = 2,3,\ldots, \\ &j = 2n: \quad m_j^o = (2n)!/\left(2^n n!\right), \quad \alpha = 0; \\ &j = 2n+1: \quad m_j^o = (2n+1)!/\left(2^n(n-1)!\right), \quad \alpha = 1 \end{aligned}\right\} \tag{14}$$

within the numerical values for the factors $m_j^o, m_j', m_j''$ up to the 19-th order to be presented in the following table 1.

It is very evident from the table that the numerical values for the factors $m_j'$ and especially for $m_j''$ in eq. (14) grow considerably for high orders of the moments. It means that the convergence of the asymptotic series (5) may fall for extreme values of the wave steepness and wave elevations. The numerical evaluation of the distribution moments $\overline{\mu}_n$ allows us to get the corresponding estimations for cumulants of the characteristic function of the distribution, and for the first eight cumulants the final results are as follows

$$\left.\begin{aligned} &\tilde{\lambda}_3 = 3\delta_\zeta\left(1 + 11\delta_\zeta^2 + 94\delta_\zeta^4\right), \quad \tilde{\lambda}_4 = 21\delta_\zeta^2\left(1 + 41\delta_\zeta^2\right); \\ &\tilde{\lambda}_5 = 317\delta_\zeta^3\left(1 + 40\delta_\zeta^2\right), \quad \tilde{\lambda}_6 = 8510\delta_\zeta^4\left(1 - 11\delta_\zeta^2\right); \\ &\tilde{\lambda}_7 = 1.17\cdot 10^5\delta_\zeta^5\left(1 - 20\delta_\zeta^2\right), \quad \tilde{\lambda}_8 = 9.83\cdot 10^5\delta_\zeta^6\left(1 + \ldots\right), \end{aligned}\right\} \tag{15}$$

where $\tilde{\lambda}_j = \lambda_\zeta/\sigma_\zeta^j$, are the normalized cumulants and dots in the brackets are used in the formulae to indicate the values of order $O\left(\delta_\zeta^n\right)$, $n \geq 2$, did not obtained in the approach.

Eq. (15) only includes the first and second approaches on the steepness $\delta_\zeta = 2\pi\sigma_\zeta/\lambda_m$; the high terms of order $O\left(\delta_\zeta^n\right)$ $n \geq 4$ would be the subject of the further analysis. The formulae in eq. (20) display that cumulant of order $j$ is proportional to the power of wave steepness of order $(j-2)$ and this result has to be considered as very important for the next reason. The attempts to obtain the high order cumulants leads to the necessarily of the including of high order nonlinear terms in the powers of the wave elevations $\zeta_w^n$.

Note that here we used the characteristic wave steepness $\delta_\zeta$ as a measure of the intensity of irregular wave motion in real sea conditions as it has been done by the authors of the publication [3]. The numerical estimations for this steepness are as follows. If we would suppose that significant wave height may achieve the extreme values of about $h_s \sim 15 \div 18$ m and use the well-known relation $h_s = 4{,}0\sigma_\zeta$, then for the irregular waves with the average wavelength about $\lambda_m \sim 150 \div 200$ m the characteristic steepness $\delta\zeta$ would be no more than 0,15…0,22.

**Table 1.** The numerical values for the factors $m_j^o, m_j', m_j''$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $m_{2n}^o$ | 1,0 | 3,0 | 15,0 | 105 | 945 | $1{,}0 \cdot 10^4$ | $1{,}4 \cdot 10^5$ | $2{,}0 \cdot 10^6$ | $3{,}5 \cdot 10^7$ |
| $m_{2n}'$ | 1,0 | 9,0 | 30,0 | 70,0 | 135 | 231 | 364 | 540 | 765 |
| $m_{2n}''$ | 19,5 | 355 | 1563 | 5419 | $1{,}6 \cdot 10^4$ | $4{,}0 \cdot 10^4$ | $9{,}05 \cdot 10^4$ | $1{,}9 \cdot 10^5$ | $3{,}6 \cdot 10^5$ |
| $m_{2n+1}^o$ | 3,0 | 30,0 | 315 | 3780 | $5{,}2 \cdot 10^4$ | $8{,}1 \cdot 10^5$ | $1{,}42 \cdot 10^7$ | $2{,}8 \cdot 10^8$ | $5{,}9 \cdot 10^9$ |
| $m_{2n+1}'$ | 12,8 | 24,7 | 43,7 | 71,6 | 110 | 162 | 228 | 311 | 413 |
| $m_{2n+1}''$ | 168 | 666 | 1884 | 4592 | $1{,}0 \cdot 10^4$ | $2{,}0 \cdot 10^4$ | $3{,}7 \cdot 10^4$ | $6{,}4 \cdot 10^4$ | $1{,}1 \cdot 10^5$ |

It is very important to verify the theoretical results (20) and for this purpose we would compare it with the independent experimental results which have been derived at the wave-generated tank of NASA Wallops Flight Center in the USA [3]. In the Center hundred samples of the irregular wave trains with different values of characteristic steepness $\delta_\zeta$ were generated and the following approximation formulae for the first eight cumulants were obtained

$$\left.\begin{array}{l} \tilde{\lambda}_3 = 4\delta_\zeta, \quad \tilde{\lambda}_4 \sim -(0{,}1 \div 0{,}4), \quad \tilde{\lambda}_5 = -17{,}5\delta_\zeta; \\ \tilde{\lambda}_6 = -127\delta_\zeta^2, \quad \tilde{\lambda}_7 = 120\delta_\zeta, \quad \tilde{\lambda}_8 = 3{,}8 \cdot 10^3 \delta_\zeta^2 \end{array}\right\}. \tag{16}$$

Some comparisons of the results (15) and (16) are shown in Fig. 3. In the Fig. 3 the dots with one sigma bars indicate the experimental results; the solid lines indicate the corresponding approximations of the experimental results (16) and the dashed lines represent the author's results (15). On the graphics the characteristic steepness $\delta_m = \delta_\zeta / 2\pi \le 0{,}04$ has also been used. The general conclusions from the result comparison are as follows:

a) The cumulants $\tilde{\lambda}_3, \tilde{\lambda}_7$ and $\tilde{\lambda}_8$ show quite good comparison of the results for the interval of the characteristic steepness $\delta_m \le 0{,}04$ (Fig. 3, *a*, *e*, *f*).

b) Absolute values of the cumulants $\tilde{\lambda}_5$ and $\tilde{\lambda}_6$ show a relatively good comparison of the results especially for the small values of the characteristic steepness $\delta_m \le 0{,}02$, but the signs of the cumulants were opposite. There is no reasonable explanation of the sign discrepancy at present (Fig. 3, *c*, *d*).

c) For the cumulant $\tilde{\lambda}_4$ (Fig. 3, *b*) there is a very different dependence on the wave steepness both for the experiments and for the theoretical estimations. Now we do not have any reasonable explanations for the discrepancy of the results. Note that the values of this cumulant are relatively small but its theoretical estimations are growing very sharply for the characteristic wave steepness $\delta_m \le 0{,}01$.

d) Finally, all the cumulants show a significant growth for high values of the characteristic wave steepness $\delta_m > 0{,}025 - 0{,}030$ which correspond to the extreme waves in real sea conditions.

## 3. THE PROBLEM OF THE CONVERGENCE

Numerical evaluations of the nonlinear correction factor according to eq. (9) have shown such phenomenon as the existence of the intervals of non-convergence of asymptotic series. These intervals depend on the three factors: the order of the approximation n, the value of characteristic wave steepness $\delta_\zeta$ and the value of wave elevations $\zeta_w$. Fig. 4 displays three samples of the convergence for the distribution function (8) for different approximations with $n = 2...12$, three values of wave elevations $\zeta_w$: a) $\zeta_w/\sigma_\zeta = 1$, b) $\zeta_w/\sigma_\zeta = 3$, c) $\zeta_w/\sigma_\zeta = 4$ and for the following values of the characteristic wave steepness $\delta_\zeta$: +—+ — $\delta_\zeta = 0{,}063$ ($h_s/\lambda_m = 1/25$), ∇—∇ — $\delta_\zeta = 0{,}075$ ($h_s/\lambda_m = 1/21$) , Δ—Δ — $\delta_\zeta = 0{,}10$ ($h_s/\lambda_m = 1/16$), ×—× — $\delta_\zeta = 0{,}126$ ($h_s/\lambda_m = 1/12$), o—o — $\delta_\zeta = 0{,}17$ ($h_s/\lambda_m = 1/9$); horizontal solid line on the graphics is the Gaussian value of the distribution.

One can see the existence of the non-convergence of asymptotic series for great values of $\zeta_w$ and $\delta_\zeta$. In general there are two limited

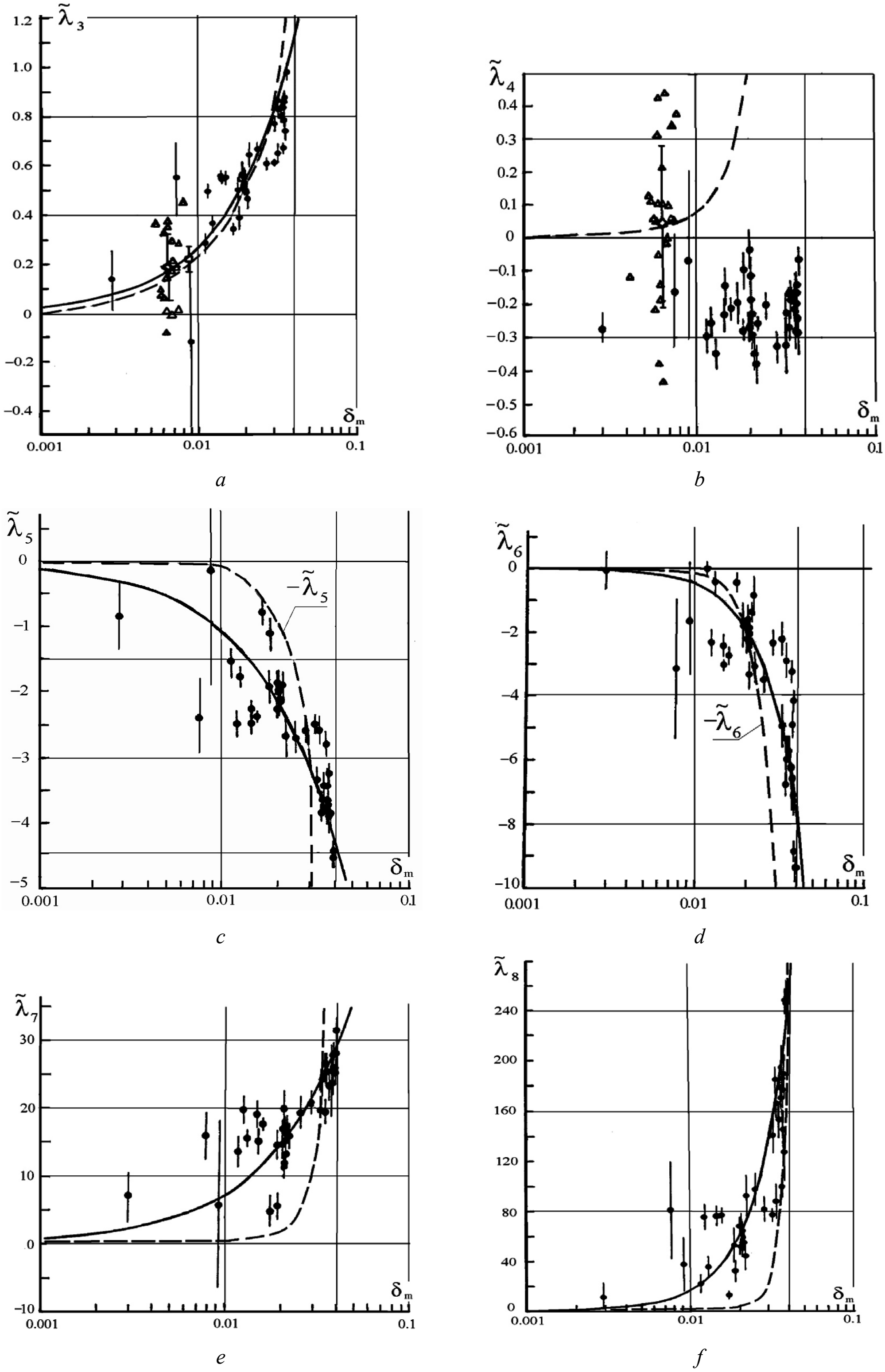
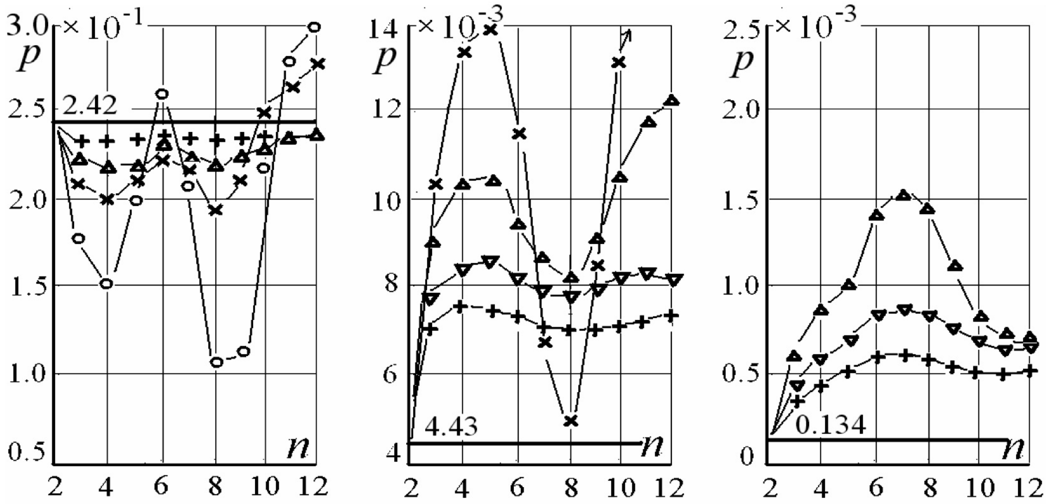



**Fig. 3.** Comparisons of the theoretical estimations for the cumulants with the experimental results [3]




*a* *b* *c*

**Fig. 4.** Convergence of the asymptotical sets for different values of wave steepness and wave elevations: $a$ — $\zeta_w/\sigma_\zeta = 1$, $b$ — $\zeta_w/\sigma_\zeta = 3$, c — $\zeta_w/\sigma_\zeta = 4$

cases in the convergence of the asymptotic series; the first one is that for the wave steepness about ($h_s/\lambda_m = 1/32$) the convergence takes place in the interval $\zeta_w/\sigma_\zeta \in [-4{,}5;\ 5{,}0]$ And the second one is that for the wave steepness ($h_s/\lambda_m = 1/9$) there is no convergence for any value of the elevations $\zeta_w$. The problem of the non-convergence of asymptotic series would be the subject of the investigations in the future.

## CONCLUSIONS

1. Nonlinear generalizations of short-term distributions of windgenerated wave elevations in extreme sea conditions by using characteristic function technique and corresponding asymptotic Gram-Edgeworht's sets can be considered as effective and useful technique. In the framework of traditional approach the asymptotical sets have been derived up to the 12-th order. 2. The explicit expressions for the cumulants of characteristic function up to the 12-th order have been derived and its numerical estimations have been done for the first eight cumulants by using nonlinear wave-group model of the 6-th order for energetic component of irregular wave motion. The results displayed that cumulants are proportional to the powers of the characteristic wave steepness $\delta_\zeta$ according to the law $\lambda_j \sim O\left(\delta_\zeta^{\,j-2}\right)$, $j \geq 3$ and for this reason we have to use high order nonlinear wave models in the estimations of high order moments and cumulants of the distribution. 3. The convergence of the asymptotical sets depends on the values of wave elevations, characteristic wave steepness and the order of the approximation and this problem would be considered in the future work.

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